SCSA Syllabus Points

1.3.1 use implication, converse, equivalence, negation, inverse, contrapositive

1.3.2 use proof by contradiction

2.3.1 prove simple results involving numbers

2.3.2 express rational numbers as terminating or eventually recurring decimals and vice versa

2.3.3 prove irrationality by contradiction for numbers such as root two

Cambridge 6A

Objectives

- To understand and use various methods of proof, including:
  - **direct proof**
  - **proof by contrapositive**
  - **proof by contradiction**.
- To write down the **negation** of a statement.
- To write and prove **converse** statements.
- To understand when mathematical statements are **equivalent**.
- To use the symbols for **implication** ($\Rightarrow$) and **equivalence** ($\Leftrightarrow$).
- To understand and use the quantifiers **'for all'** and **'there exists'**.
- To disprove statements using **counterexamples**.
- To understand and use the **principle of mathematical induction**.

$\forall$ for all

$\exists$ there exists

Mathematics is special

- A mathematical proof is an argument that demonstrates the absolute truth of a statement
- One can only "prove" in Mathematics – this certainty makes Mathematics different from other sciences
- In science, a theory is never proved true, instead, the aim is to prove that a theory is not true (falsification, ref. Karl Popper)
- Even if falsifying evidence is hard to find, this increases likelihood the theory is correct, but not a guarantee.

A proof should be

~ correct

~ clear

~ simple

ref. Occam's razor

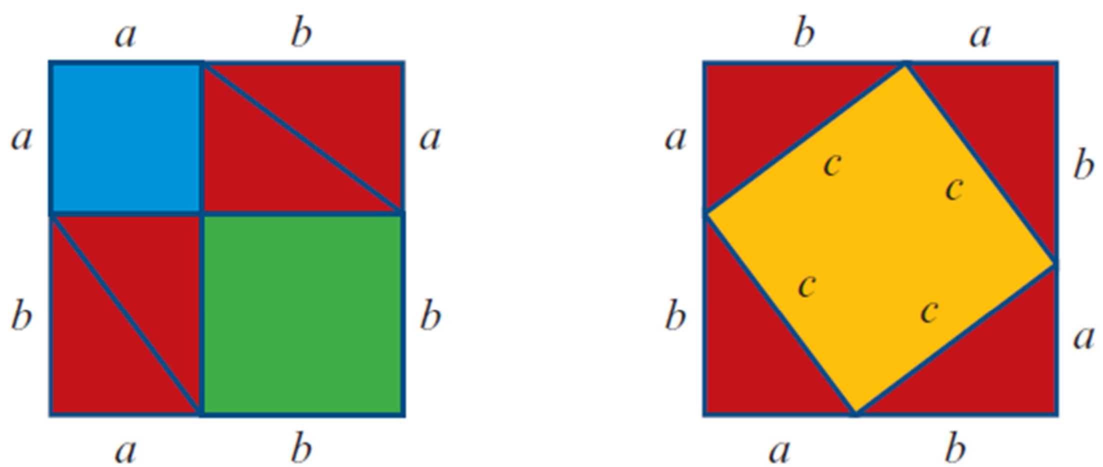
Pythagoras' theorem

Take any triangle with side lengths a, b and c. If the angle between a and b is 90°, then

$$a^2 + b^2 = c^2$$

Proof Consider the two squares shown below.

also,
$(a+b)^2 =$
$a^2 + 2ab + b^2$



The two squares each have the same total area. So subtracting four red triangles from each figure will leave the same area. Therefore $a^2 + b^2 = c^2$.

an elegant proof

6A Direct proof

$\Rightarrow$ is implication

▶ Conditional statements

Consider the following sentence:

Statement	If	it is raining	then	the grass is wet.

This is called a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

hypothesis conclusion

This can be abbreviated as

$$P \Rightarrow Q$$

which is read 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

Not all conditional statements will be true. For example, switching the hypothesis and the conclusion above gives:

Statement	If	the grass is wet	then	it is raining.

what else can cause the grass to be wet?

Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

assume a is odd & b is even

since a is odd, $a = 2m + 1, \forall m \in \mathbb{Z}$

since b is even, $b = 2n, \forall n \in \mathbb{Z}$

use m & n as a & b may be different

$$\begin{aligned}\therefore a + b &= (2m+1) + 2n \\ &= 2m + 2n + 1 \\ &= 2(m+n) + 1 \\ &= 2k + 1 \quad \text{where } k = (m+n) \in \mathbb{Z}\end{aligned}$$

hence $a+b$ is odd

Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

assume both a and b are odd

thus $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$

$\therefore$ $ab = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$ write it in this format

$= 2k + 1$ where $k = (2mn + m + n) \in \mathbb{Z}$

hence ab is odd

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Since p is divisible by 5, $p = 5m$ for some $m \in \mathbb{Z}$

since q is divisible by 3, $q = 3n$ for some $n \in \mathbb{Z}$

thus $pq = (5m)(3n)$

$= 15mn$ ← a multiple of 15...

$\therefore$ pq is divisible by 15

Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

assume $x > y$, meaning $x - y > 0$

since x & y are positive, $x + y > 0$

$\therefore \quad x^2 - y^2 = (x-y)(x+y) > 0$

+ve +ve

Hence $x^2 > y^2$

easier to prove

$x^2 - y^2 > 0$

due to D.O.T.S

$x^2 - y^2$ is more versatile

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

implies

$$\frac{x+y}{2} \geq \sqrt{xy}$$

← assume this is true & hopefully end up with a correct statement!

$$\Rightarrow x+y \geq 2\sqrt{xy}$$

$$\Rightarrow (x+y)^2 \geq 4xy$$

true

$$\Rightarrow x^2+2xy+y^2 \geq 4xy$$

$$\Rightarrow x^2-2xy+y^2 \geq 0$$

$$(x-y)^2 \geq 0$$

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

implies

$$\frac{x+y}{2} \geq \sqrt{xy}$$

true?

assume

$$\Rightarrow \quad x+y \geq 2\sqrt{xy}$$

$$\Rightarrow \quad (x+y)^2 \geq 4xy$$

true but we can not

$$\Rightarrow \quad x^2+2xy+y^2 \geq 4xy$$

$$\Rightarrow \quad x^2-2xy+y^2 \geq 0$$

$$(x-y)^2 \geq 0$$

luckily we can reverse the steps & use $a > b \Rightarrow \sqrt{a} > \sqrt{b}$

Breaking a proof into cases

Sometimes it helps to break a problem up into different cases.

Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'We are both knaves.' What are Alice and Bob?

Breaking a proof into cases

Sometimes it helps to break a problem up into different cases.

Alice spoke

cant be mixed as this is partially true

Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'We are both knaves.' What are Alice and Bob?

Suppose Alice is a knight
$\Rightarrow$ Alice is telling the truth
$\Rightarrow$ Alice & Bob are both knaves
$\Rightarrow$ Alice is a knave & a knight
which is impossible.

Suppose Alice is a knave
$\Rightarrow$ Alice is lying
$\Rightarrow$ Alice & Bob are both not knaves
$\Rightarrow$ Bob is a knight

$\therefore$ Alice must be a knave & Bob a knight

Assigned Task

- **Cambridge Specialist Maths:**

- **Exercise** **6A** (18 Qs)