Defining the scalar product (dot product)

The scalar product is an operation that takes two vectors and gives a real number.

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Definition of the scalar product

We define the **scalar product** of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$$

product of vertical component magnitudes and horizontal component magnitudes

$\underline{a} = 2\underline{i} + 3\underline{j}$; $\underline{b} = \underline{i} - 4\underline{j}$

$\underline{a} \cdot \underline{b} = (2\times1) + (3\times-4)$

$= -10$

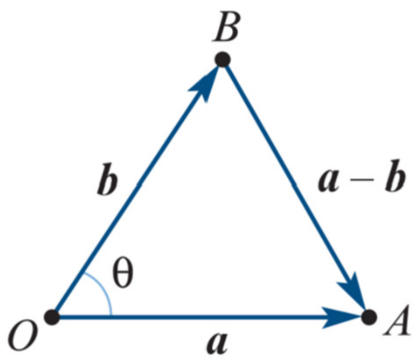
$\underline{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; $\underline{b} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$\underline{a} \cdot \underline{b} = (2\times1) + (3\times-4)$

$= -10$

but what does this number mean?

Consider the vectors below



let $\underline{a} = a_1\underline{i} + a_2\underline{j}$ & $\underline{b} = b_1\underline{i} + b_2\underline{j}$

$$|\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\theta = |\underline{a} - \underline{b}|^2$$

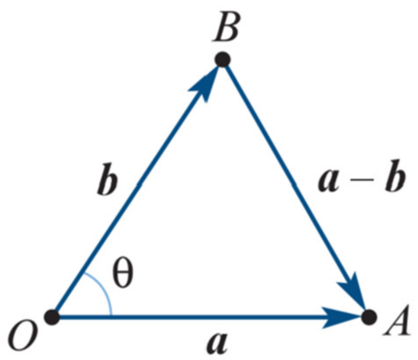
$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|a||b|\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$(\underline{a} - \underline{b}) = (a_1\underline{i} + a_2\underline{j}) - (b_1\underline{i} + b_2\underline{j})$$

$$= a_1\underline{i} - b_1\underline{i} + a_2\underline{j} - b_2\underline{j}$$

$$= (a_1 - b_1)\underline{i} + (a_2 - b_2)\underline{j}$$

Consider the vectors below



$$\underline{a} - \underline{b} = a_1\underline{i} - b_1\underline{i} - a_2\underline{j} - b_2\underline{j}$$

let $\underline{a} = a_1\underline{i} + a_2\underline{j}$ & $\underline{b} = b_1\underline{i} + b_2\underline{j}$

$$|\underline{a}|^2 + |\underline{b}|^2 - 2|\underline{a}||\underline{b}|\cos\theta = |\underline{a} - \underline{b}|^2$$

$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|a||b|\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - (a_1 - b_1)^2 - (a_2 - b_2)^2 = 2|a||b|\cos\theta$$

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 - [a_1^2 - 2a_1b_1 + b_1^2] - [a_2^2 - 2a_2b_2 + b_2^2] =$$

$$a_1^2 + a_2^2 + b_1^2 + b_2^2 - a_1^2 + 2a_1b_1 - b_1^2 - a_2^2 + 2a_2b_2 - b_2^2 =$$

$$2a_1b_1 + 2a_2b_2 = 2|a||b|\cos\theta$$

$$a_1b_1 + a_2b_2 = |a||b|\cos\theta$$

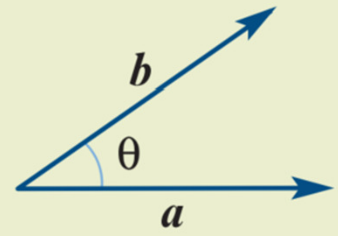
$a_1b_1 + a_2b_2$
was -10 in our example. It allows us to quickly calculate

Geometric description of the scalar product

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

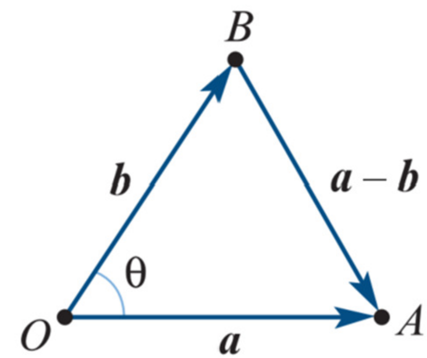
where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



So $a_1 b_1 + a_2 b_2 = |a||b| \cos\theta$

is the operation

$\underline{a} \cdot \underline{b} = |a||b| \cos\theta$



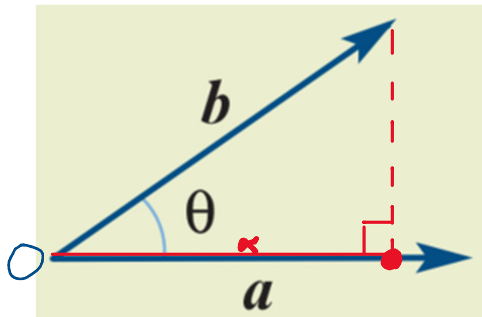
notice that if $\theta = 90°$ (or $\frac{\pi}{4}$ radians), $\underline{a} \cdot \underline{b}$ has its maximum value

and if $\theta = 0°$, $\underline{a} \cdot \underline{b} = 0$

so $\underline{a} \cdot \underline{b}$ measures how much two vectors point in the same direction (in practical terms)

or - how close they are to being parallel.

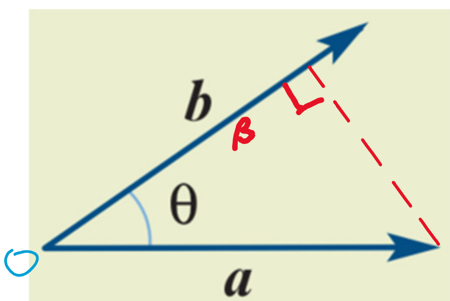
$$\cos\theta = \frac{|\alpha|}{|\underline{b}|} \longrightarrow |\alpha| = |b| \cos\theta$$



let $\underline{a}$ be the displacement of object o.
the component of force $\underline{b}$ acting in the same direction as displacement is $|b| \cdot \cos\theta$

$= |\alpha| |a|$

$\therefore$ Work $= |b| \cos\theta \times |a|$
$= |a| |b| \cos\theta$



$$\cos\theta = \frac{|\beta|}{|a|} \longrightarrow |\beta| = |a| \cos\theta$$

If force $\underline{a}$ acts on O & displacement is along $|b|$, work $= |a| \cos\theta \times |b|$
$= |a| |b| \cos\theta$

$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$

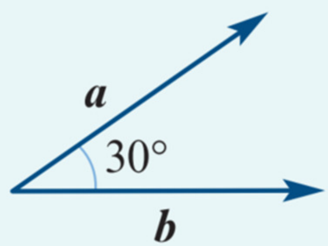
Example 10

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

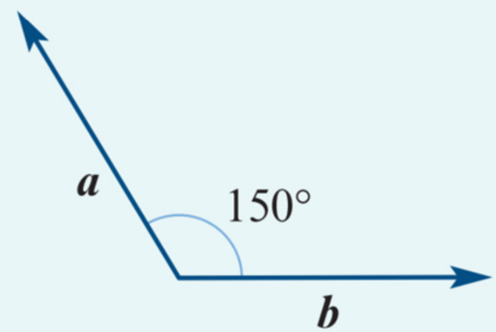
b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 30° \\ &= 20 \times \frac{\sqrt{3}}{2} \\ &= 10\sqrt{3}\end{aligned}$$



b
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= 4 \times 5 \times \cos 150° \\ &= 20 \times \frac{-\sqrt{3}}{2} \\ &= -10\sqrt{3}\end{aligned}$$



Properties of the scalar product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \mathbf{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$

Finding the magnitude of the angle between two vectors

$\underline{a} \cdot \underline{b} = |a||b| \cos\theta \longrightarrow \cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|a||b|}$

$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2$

$\therefore \quad |a||b| \cos\theta = a_1 b_1 + a_2 b_2$

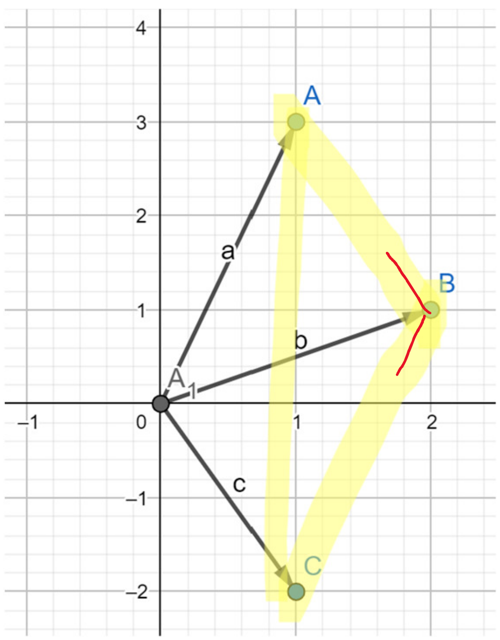
$\cos\theta = \dfrac{a_1 b_1 + a_2 b_2}{|\underline{a}||\underline{b}|} = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$

Example 11

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \quad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$$

Find the magnitude of $\angle ABC$.



$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

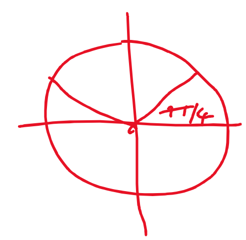
$$\overrightarrow{BA} = -\underline{b} + \underline{a} = \underline{a} - \underline{b} = -\underline{i} + 2\underline{j}$$

$$\overrightarrow{BC} = -\underline{b} + \underline{c} = \underline{c} - \underline{b} = -\underline{i} - 3\underline{j} \qquad \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -3 \end{bmatrix} = -5$$

$$|\overrightarrow{BA}| = \sqrt{1+4} = \sqrt{5} \; ; \; |\overrightarrow{BC}| = \sqrt{1+9} = \sqrt{10}$$

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = -\frac{1}{\sqrt{2}}$$



$$\therefore \angle ABC = \frac{3\pi}{4}$$