Rossmoyne Senior High School

Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	91	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

SN085-191-3

Section One: Calculator-free 35% (50 Marks)

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 (6 marks)

C_1 and C_2 are chords in the same circle. Let P be the statement 'chords C_1 and C_2 have equal lengths' and Q be the statement 'chords C_1 and C_2 subtend equal angles at the centre'.

(a) Write, in words, the negation of P. (1 mark)

Solution
Chords C_1 and C_2 do not have equal lengths.
Specific behaviours
✓ correct statement, in words

(b) Write, in words, the meaning of $P \Rightarrow Q$. (1 mark)

Solution
If chords C_1 and C_2 have equal lengths, then chords C_1 and C_2 subtend equal angles at the centre.
Specific behaviours
✓ correct statement, in words

(c) Write, symbolically, the converse of $P \Rightarrow Q$. (1 mark)

Solution
$Q \Rightarrow P$
Specific behaviours
✓ correct statement, in symbols

(d) Write, in any form, the inverse of $P \Rightarrow Q$. (1 mark)

Solution
$\neg P \Rightarrow \neg Q$; or 'If not P then not Q'; or 'If chords C_1 and C_2 do not have equal lengths, then chords C_1 and C_2 do not subtend equal angles at the centre'. NB: Accept other notations for 'Not', eg P' or ~P
Specific behaviours

(e) Write, in any form, the contrapositive of $P \Rightarrow Q$ and state, with justification, whether the contrapositive is true. (2 marks)

Solution
$\neg Q \Rightarrow \neg P$; or 'If not Q then not P'; or 'If chords C_1 and C_2 do not subtend equal angles at the centre, then chords C_1 and C_2 do not have equal lengths'. The contrapositive statement is true, as it is always logically equivalent to the original statement. NB: Accept other notations for 'Not', eg P' or ~P
Specific behaviours
✓ correct statement ✓ states true, with justification

Question 2 **(5 marks)**

Use the inclusion-exclusion principle to determine how many integers between 1 and 181 inclusive are divisible by $3, 5$ or 10.

Solution
Divisible by 3; or by 5; or by 10: $\lfloor 181 \div 3 \rfloor = 60$ $\lfloor 181 \div 5 \rfloor = 36$ $\lfloor 181 \div 10 \rfloor = 18$ Divisible by 3 and 5 (15); or by 3 and 10 (30); or by 5 and 10 (10): $\lfloor 181 \div 15 \rfloor = 12$ $\lfloor 181 \div 30 \rfloor = 6$ $\lfloor 181 \div 10 \rfloor = 18$ Divisible by 3 and 5 and 10 (30): $\lfloor 181 \div 30 \rfloor = 6$ Using the inclusion-exclusion principle: $n = 60 + 36 + 18 - 12 - 6 - 18 + 6$ $= 114 - 36 + 6$ $= 84$
Specific behaviours
✓ clearly indicates methodical approach ✓ at least two correct cases for divisible singly ✓ at least two correct cases for divisible by pairs ✓ correct number for divisible by all three ✓ obtains correct number of integers, no errors *(NB No marks for correct answer if no evidence of use of inclusion-exclusion principle)*

Question 3 **(8 marks)**

Let $\mathbf{a} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(a) Determine

(i) $2\mathbf{a} + 3\mathbf{c}$. (2 marks)

Solution
$2\mathbf{a} + 3\mathbf{c} = 2\begin{pmatrix} 7 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} 14 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 6 \end{pmatrix}$ $= \begin{pmatrix} 11 \\ 10 \end{pmatrix}$
Specific behaviours
✓ correct multiples ✓ correct vector

(ii) $|\mathbf{b} - \mathbf{a}|$. (2 marks)

Solution
$\mathbf{b} - \mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$ $\lvert\mathbf{b} - \mathbf{a}\rvert = \sqrt{25 + 4} = \sqrt{29}$
Specific behaviours
✓ difference of vectors ✓ correct magnitude

(b) Given that $\mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c}$, determine the value of the constant λ and the value of the constant μ. (4 marks)

Solution
$\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ Equating $\mathbf{i}, \mathbf{j}$ coefficients $2\lambda - \mu = 7$ $4\lambda + 2\mu = 2 \rightarrow 2\lambda + \mu = 1$ Add equations $4\lambda = 8 \rightarrow \lambda = 2$ Substitute $\mu = 2\lambda - 7 = 4 - 7 = -3$ Hence $\lambda = 2, \qquad \mu = -3$
Specific behaviours
✓ equates i-coefficients ✓ equates i-coefficients ✓ solves for one constant ✓ both correct constants

Question 4 **(7 marks)**

Two vectors are $\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$.

(a) Determine the vector projection of $\mathbf{a}$ on $\mathbf{b}$. (3 marks)

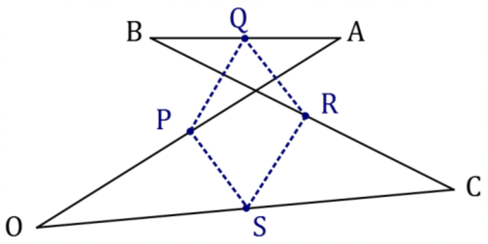
Solution
$\dfrac{\begin{pmatrix}6\\8\end{pmatrix}\cdot\begin{pmatrix}3\\-1\end{pmatrix}}{\begin{pmatrix}3\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\end{pmatrix}}\begin{pmatrix}3\\-1\end{pmatrix} = \dfrac{10}{10}\begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$
Specific behaviours
✓ indicates correct method ✓ correct scalar products ✓ correct vector

(b) Vector $\mathbf{c} = x\mathbf{i} + y\mathbf{j}$ has the same magnitude as $\mathbf{a}$ and is perpendicular to $\mathbf{b}$. Determine the values of the constants x and y. (4 marks)

Solution
Require $\begin{pmatrix}x\\y\end{pmatrix}\cdot\begin{pmatrix}3\\-1\end{pmatrix} = 0$ and $\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\| = \left\|\begin{pmatrix}6\\8\end{pmatrix}\right\|$. Hence $3x - y = 0$ and $x^2 + y^2 = (6^2 + 8^2) = 100$. $x^2 + (3x)^2 = 100$ $10x^2 = 100$ $x = \pm\sqrt{10}, \quad y = 3x$ Hence $x = \sqrt{10}, y = 3\sqrt{10}$ or $x = -\sqrt{10}, y = -3\sqrt{10}$. NB: If x and y are not given as paired solutions, max 3 marks. Eg $x = \pm\sqrt{10}, y = \pm 3\sqrt{10}$
Specific behaviours
✓ equation using perpendicular ✓ equation using magnitude ✓ solves for one constant ✓ states both solution sets

Question 5 **(8 marks)**

Crossed quadrilateral $OABC$ is shown in the diagram.



The midpoints of sides OA, AB, BC and CO are P, Q, R and S respectively.

Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(a) Draw quadrilateral $PQRS$ on the diagram. (1 mark)

(b) Determine expressions for $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ and $\overrightarrow{OS}$ in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. (3 marks)

Solution (a) (b)
(a) see diagram for $PQRS$. (b) $\overrightarrow{OP} = \frac{1}{2}\mathbf{a}, \quad \overrightarrow{OS} = \frac{1}{2}\mathbf{c}$ $\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ $\overrightarrow{OR} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB} = \mathbf{c} + \frac{1}{2}(\mathbf{b} - \mathbf{c}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$
Specific behaviours
✓ (a) correct midpoints, joined to form $PQRS$ ✓ expressions for $\overrightarrow{OP}, \overrightarrow{OS}$ ✓ expression for $\overrightarrow{OQ}$ ✓ expression for $\overrightarrow{OR}$

(c) Prove that the midpoints of the sides of a crossed quadrilateral join to form a parallelogram. (4 marks)

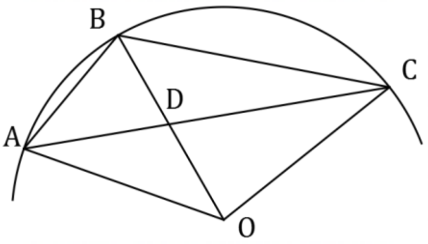
Solution
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{b}$ $\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{c} = \frac{1}{2}\mathbf{b}$ Since $\overrightarrow{PQ} = \overrightarrow{SR}$ then $PQRS$ has a pair of congruent and parallel sides and is a parallelogram. Hence, the midpoints of the sides of a crossed quadrilateral join to form a parallelogram. *(NB can use other pair of sides, different reasoning, etc)* *In this case, the vector is ½(**a** – **c**)*
Specific behaviours
✓ expression for $\overrightarrow{PQ}$ or $\overrightarrow{QR}$ ✓ expression for $\overrightarrow{SR}$ or $\overrightarrow{PS}$ ✓ correctly reasons that $PQRS$ is a parallelogram ✓ completes proof

Question 6 **(8 marks)**

(a) Points A, B and C lie on an arc of a circle with centre O as shown at right.

Chord AC intersects OB at point D.

The diagram is not drawn to scale.



When $\angle ABC = 128°$ and $\angle BAC = 28°$, determine the size of $\angle ADB$. (4 marks)

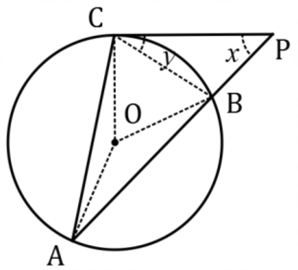
Solution
$\angle BOC = 2\angle BAC = 2 \times 28° = 56°$ $\angle BCA = 180° - 128° - 28° = 24°$ $\angle OBC = \frac{1}{2}(180° - 56°) = 62°$ $\angle ADB = 62° + 24° = 86°$
Specific behaviours
✓ obtains $\angle BOC$ ✓ obtains $\angle BCA$ ✓ obtains $\angle OBC$ ✓ correct $\angle ADB$

Alternate Solution
$\angle BCA = 180° - 128° - 28° = 24°$ $\angle BOA = 2 \times \angle BCA = 48°$ $\angle OBA = \frac{1}{2}(180° - 48°) = 66°$ $\angle ADB = 180 - 66° - 28° = 86°$
Specific behaviours
✓ obtains $\angle BCA$ ✓ obtains $\angle BOA$ ✓ obtains $\angle OBA$ ✓ correct $\angle ADB$

(b) A secant cuts a circle with centre O at points A and B. Secant AB is extended beyond B to point P, where it meets a line that is a tangent to the circle at point C. Prove that the size of $\angle CPB$ is equal to one half the difference of the sizes of $\angle AOC$ and $\angle BOC$. (4 marks)

Solution

Let $\angle CPB = x$ and $\angle PCB = y$. Then



$$\angle ABC = x + y \quad (1)$$
$$\angle AOC = 2x + 2y \quad (2)$$
$$\angle BAC = y \quad (3)$$
$$\angle BOC = 2y \quad (2)$$

Hence

$$\frac{1}{2}(\angle AOC - \angle BOC) = \frac{1}{2}(2x + 2y - 2y)$$
$$= x$$
$$= \angle CPB$$

Reasoning:
(1) sum of two interior angles is equal to the opposite exterior angle
(2) angle at centre property
(3) angle in alternate segments
NB: Alternate Solution on page 10

Specific behaviours

✓ reasonable diagram (variations exist - secant between O and C, etc.)
✓ derives expression for $\angle AOC$
✓ derives expression for $\angle BOC$
✓ completes proof, with reasonable explanation throughout

Question 7 **(8 marks)**

(a) Determine all vectors $\mathbf{u}$ that meet the following two conditions: (3 marks)

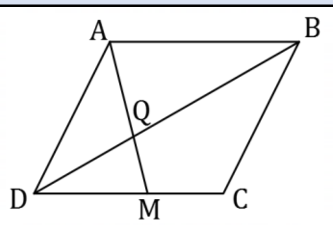
$$(3\mathbf{i}+4\mathbf{j})\bullet\mathbf{u}=0 \text{ and } \mathbf{u}\bullet\mathbf{u}=1$$

Solution
$\mathbf{u}\bullet\mathbf{u}=1 \Rightarrow \boldsymbol{u}=\widehat{\boldsymbol{u}}$ (ie $\mathbf{u}$ is a unit vector) $\boldsymbol{u}=-\frac{4}{5}\boldsymbol{i}+\frac{3}{5}\mathbf{j}$ ***OR*** $\boldsymbol{u}=\frac{4}{5}\boldsymbol{i}-\frac{3}{5}\mathbf{j}$
Specific behaviours
✓ Vector meets condition 1 (vector is perpendicular to $3\mathbf{i}+4\mathbf{j}$) ✓ Vector meets condition 2 (ie a unit vector) ✓ Finds both solutions.

(b) $ABCD$ is a parallelogram and M is the midpoint of DC. Diagonal DB intersects AM at Q so that $\overrightarrow{AQ}=h\overrightarrow{AM}$ and $\overrightarrow{DQ}=k\overrightarrow{DB}$. Use a vector method to determine the value of the constant h and the value of the constant k. (5 marks)

Solution

$$\begin{aligned}\overrightarrow{DQ}&=\overrightarrow{DA}+\overrightarrow{AQ}\\&=\overrightarrow{DA}+h\overrightarrow{AM}\\&=\overrightarrow{DA}+h\left(\frac{1}{2}\overrightarrow{DC}-\overrightarrow{DA}\right)\\&=(1-h)\overrightarrow{DA}+\frac{h}{2}\overrightarrow{DC}\quad(1)\end{aligned}$$



But also

$$\begin{aligned}\overrightarrow{DQ}&=k\overrightarrow{DB}\\&=k(\overrightarrow{DA}+\overrightarrow{DC})\\&=k\overrightarrow{DA}+k\overrightarrow{DC}\quad(2)\end{aligned}$$

Equating (1) and (2)

$$k\overrightarrow{DA}+k\overrightarrow{DC}=(1-h)\overrightarrow{DA}+\frac{h}{2}\overrightarrow{DC}$$

Equating vector coefficients

$$k=1-h \text{ and } k=\frac{h}{2}\Rightarrow\frac{h}{2}=1-h\Rightarrow h=\frac{2}{3},\qquad k=\frac{1}{3}$$

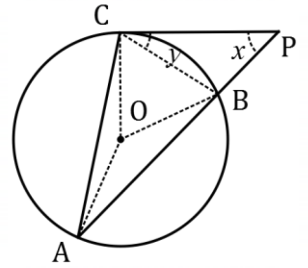
Specific behaviours

✓ sketch diagram
✓ expresses $\overrightarrow{DQ}$ in terms of $\overrightarrow{DA}$ and $\overrightarrow{AM}$
✓ obtains equation (1)
✓ obtains equation (2)
✓ equates coefficients to obtain value of each constant
(*Many variations exist using different sides or* $\overrightarrow{DA}=\mathbf{a}$, *etc*)

End of questions

Supplementary page

Question number: _________

Alternate Solution to Q6b
$\frac{1}{2}(\angle AOC - \angle BOC) = \frac{1}{2}\angle AOC - \frac{1}{2}\angle BOC$ $= \angle ABC - \angle BAC \quad (1)$ $= (180 - \angle ACP) - \angle PAC \quad (2)$ $= 180 - (\angle ACP + \angle PAC)$ $= \angle CPB \quad (3)$  Reasoning: (1) central angle theorem (2) alternate segment theorem (3) sum of angles in a triangle
Specific behaviours
✓ reasonable diagram (variations exist - secant between O and C, etc.) ✓ uses central angle theorem to get (1) ✓ uses alternate segment theorem to get (2) ✓ completes proof, with reasonable explanation throughout

Supplementary page

Question number: _________