Rossmoyne Senior High School

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators, which can include scientific, graphic and Computer Algebra System (CAS) calculators, are permitted in this ATAR course examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	91	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (91 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

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Question 8 **(8 marks)**

Let $\underset{\sim}{a} = \begin{pmatrix} 4 \\ k-1 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} k \\ 3 \end{pmatrix}$. Determine

(a) $\underset{\sim}{a} - \underset{\sim}{b}$ when $k = 5$. (1 mark)

Solution
$\begin{pmatrix} 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
Specific behaviours
✓ correct vector

(b) the value(s) of k for which $\left|\underset{\sim}{b}\right| = 3\sqrt{5}$. (2 marks)

Solution
$\left|\begin{pmatrix} k \\ 3 \end{pmatrix}\right| = 3\sqrt{5} \Rightarrow k^2 + 9 = 45$ $k^2 = 36, \qquad k = \pm 6$
Specific behaviours
✓ forms equation using magnitude ✓ correct values of k

(c) the value(s) of k for which $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular. (2 marks)

Solution
$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \Rightarrow 4k + 3(k-1) = 0$ $7k - 3 = 0, \qquad k = \frac{3}{7}$
Specific behaviours
✓ indicates scalar product is 0 (mark **can't** be given just for writing $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$) ✓ correct value of k

(d) the value(s) of k for which $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel. (3 marks)

Solution
$\underset{\sim}{a} = \mu \underset{\sim}{b} \Rightarrow \mu = \frac{4}{k} = \frac{k-1}{3}$ $k^2 - k - 12 = 0$ $(k-4)(k+3) = 0$ $k = 4, \qquad k = -3$
Specific behaviours
✓ forms equation ✓ one value of k ✓ all correct values of k

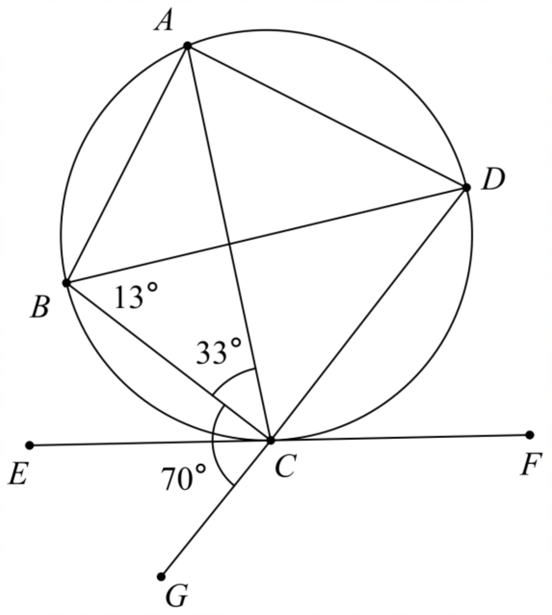
Alternate Solution
$\underset{\sim}{a} = \lambda \underset{\sim}{b} \Rightarrow \begin{pmatrix} 4 \\ k-1 \end{pmatrix} = \lambda \begin{pmatrix} k \\ 3 \end{pmatrix}$ $\Rightarrow \begin{cases} 4 = k \times \lambda \\ k - 1 = 3\lambda \end{cases}$ Using ClassPad to solve, $k = 4, \qquad k = -3$
Specific behaviours
✓ forms simultaneous equations by matching components ✓ one value of k ✓ all correct values of k

Question 9 **(6 marks)**

$ABCD$ is a cyclic quadrilateral.

EF is a tangent at C, and DCG is a straight line.

$\angle BCA = 33°$
$\angle GCB = 70°$
$\angle DBC = 13°$



(a) Determine the following angles, giving reasons

(i) $\angle BAD$ (2 marks)

Solution
$\angle BCD = 110°$ (angles on straight line) $\angle BAD = 70°$ (opp. $\angle's$ cyclic quad. are suppl.)
Specific behaviours
✓ determines $\angle BAD$ ✓ gives appropriate reason(s)

(ii) $\angle BDA$ (1 marks)

Solution
$\angle BDA = 33°$ ($\angle's$ in the same segment equal)
Specific behaviours
✓ determines $\angle BDA$ with appropriate reason

(b) Prove that AC passes through the centre of the circle, justifying your answer. (3 marks)

Solution
$\angle ABD = 180 - \angle BAD - \angle BDA$ (triangle angle sum) $= 180° - 70° - 33° = 77°$ $\Rightarrow \angle ABC = 90°$ Hence, AC must be the diameter as angles in a semicircle are $90°$, so AC passes through the centre of the circle
Specific behaviours
✓ determines $\angle ABD$ with reasons ✓ determines $\angle ABC$ ✓ explains why AC is the diameter and concludes centre lies on AC

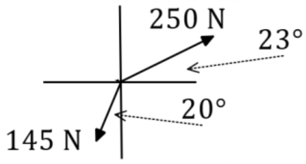
Question 10 **(8 marks)**

Two forces act on a particle. One force of 145 N acts on a bearing of $200°T$ and another force of 250 N acts on a bearing of $067°T$.

Let the perpendicular unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$ act in an easterly and northerly direction respectively.

In this question give coefficients correct to 2 decimal places.

(a) Sketch a neat, clearly labelled diagram of the two forces acting on the particle showing all magnitudes and angles. (1 mark)

Solution

Specific behaviours
✓ correct forces and angles

(b) Express the 250 N force in component form using the unit vectors $\underset{\sim}{i}$ and $\underset{\sim}{j}$. (2 marks)

Solution
Force of 250 N acts at an angle of $90° - 67° = 23°$ to positive x-axis: $$\underset{\sim}{f_1} = \begin{pmatrix} 250\cos(23°) \\ 250\sin(23°) \end{pmatrix} = 230.13\underset{\sim}{i} + 97.68\underset{\sim}{j}$$
Specific behaviours
✓ indicates correct use of trigonometry to convert ✓ correct coefficients in any component form (Ans only ok)

(c) Determine the resultant of the two forces, leaving your answer in component form. (3 marks)

Solution
$$\underset{\sim}{f_2} = \begin{pmatrix} 145\cos(90° - 200°) \\ 145\sin(90° - 200°) \end{pmatrix} = \begin{pmatrix} -49.59 \\ -136.26 \end{pmatrix}$$ $$\underset{\sim}{R} = \begin{pmatrix} 230.13 \\ 97.68 \end{pmatrix} + \begin{pmatrix} -49.59 \\ -136.26 \end{pmatrix} = \begin{pmatrix} 180.53 \\ -38.57 \end{pmatrix}$$ Resultant is $\begin{pmatrix} 180.53 \\ -38.57 \end{pmatrix}$ N.
Specific behaviours
✓ converts 145 N force into component form ✓ shows resultant as a sum of the two forces ✓ correct resultant in component form with correct rounding

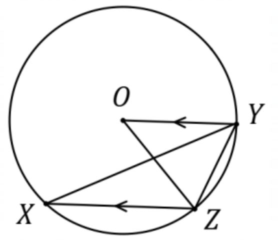
(d) A third force of 215 N is added to these existing two forces, so there are now three forces acting on the same particle. Can the three forces be brought to a state of equilibrium? Justify your answer. (2 marks)

Solution
No. The resultant of the other two forces is: $\underset{\sim}{R} = \begin{pmatrix} 180.53 \\ -38.57 \end{pmatrix}$ $\left\lvert \underset{\sim}{R} \right\rvert = \sqrt{180.53^2 + 38.57^2} = 184.61 \text{ N}$ For equilibrium, the force of 215 N would have to be equal in magnitude and opposite in direction to $\underset{\sim}{R}$ but since $\left\lvert \underset{\sim}{R} \right\rvert = 184.61 \neq 215$ N, this can never be the case.
Specific behaviours
✓ calculates magnitude of resultant ✓ states no, with reasoning

Alternate Solution
No. The resultant of the other two forces is: $\underset{\sim}{R} = \begin{pmatrix} 180.53 \\ -38.57 \end{pmatrix}$ Suppose the 215 N force acts on an angle of $\theta°$ (where $90° < \theta < 180°$), measured anti-clockwise from the positive x-axis. This can then be expressed in component form: $\underset{\sim}{f_3} = \begin{pmatrix} 215\cos\theta° \\ 215\sin\theta° \end{pmatrix}$ For equilibrium, the force of 215 N would have to be equal to the opposite of $\underset{\sim}{R}$ $\Rightarrow -\begin{pmatrix} 215\cos\theta° \\ 215\sin\theta° \end{pmatrix} = \begin{pmatrix} 180.53 \\ -38.57 \end{pmatrix}$ From $\boldsymbol{i}$ component: $-215\cos\theta° = 180.53 \quad \Rightarrow \quad \theta° = 147°$ From $\boldsymbol{j}$ component: $-215\sin\theta° = -38.57 \quad \Rightarrow \quad \theta° = 10°, 170°$ Hence, since the values of $\theta°$ solved are not the same, it is not possible for the three forces to act in equilibrium.
Specific behaviours
✓ sets up the two equations and solves for different values of $\theta°$ ✓ states no, with reasoning

Question 11 **(9 marks)**

(a) The diagram shows a circle, centre O, and points X, Y and Z that lie on it. OY is parallel to XZ.

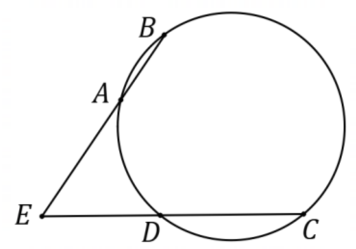


Determine the size of $\angle OZY$ and the size of $\angle XZY$ when $\angle OYX = 26°$.

Reasons are not required. (3 marks)

Solution
$\angle YXZ = \angle OYX = 26°$
$\angle YOZ = 2 \times \angle YXZ = 2 \times 26° = 52°$
$\angle OZY = (180° - \angle YOZ) \div 2 = (180° - 52°) \div 2 = 64°$
$\angle XZY = 180° - \angle OYZ = 180° - 64° = 116°$ $(\angle OYZ = \angle OZY)$
Specific behaviours
✓ correct $\angle YOZ$
✓ correct $\angle OZY$
✓ correct $\angle XZY$

(b) In the diagram, secants AB and CD of a circle intersect at E.



(i) Prove that $EA \times EB = ED \times EC$. (4 marks)

Solution
Construct lines AD and BC. $\angle ADC = 180 - \angle ADE$ (straight line) $\Rightarrow \angle ABC = 180 - \angle ADC$ (opp. ∠'s cyclic quad. are suppl.) $= \angle ADE$ Considering ΔADE and ΔCBE, $\angle AED = \angle CEB$ (common) $\angle ABC = \angle ADE$ (as above) Hence, by AA, $\Delta ADE \sim \Delta CBE$. $\Rightarrow \frac{EA}{EC} = \frac{ED}{EB}$ (corresponding sides of similar triangles) $\therefore EA \times EB = ED \times EC$ as required.
Specific behaviours
✓ proves $\angle ABC = \angle ADE$ with reasons (or $\angle BCE = \angle DAE$) ✓ states $\angle AED = \angle CEB$ with reasons ✓ concludes $\Delta ADE \sim \Delta CBE$ by AA ✓ shows how $EA \times EB = ED \times EC$ results from equal ratios

(ii) Determine the length AB when $AE = 6$ cm, $DE = 5$ cm and $DC = 4$ cm. (2 marks)

Solution
$EC = 4 + 5 = 9, \quad AB = x, \quad EB = 6 + x$ Substituting into $EA \times EB = ED \times EC$: $6(6 + x) = 5 \times 9$ $x = 1.5$ Hence $AB = 1.5$ cm.
Specific behaviours
✓ correct use of intersecting secant theorem ✓ correct length with units

Question 12 **(7 marks)**

(a) An unordered selection of 5 dishes must be chosen from 7 cold and 6 hot dishes. Determine the number of ways the selection can be made if

(i) there are no restrictions. (1 mark)

Solution
$\binom{13}{5} = 1287$ ways.
Specific behaviours
✓ correct number of ways

(ii) there must be at least one of each type and more cold than hot dishes. (2 marks)

Solution
$(c, h) = (3, 2)$ or $(4, 1)$ $\binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} = 35 \times 15 + 35 \times 6 = 525 + 210 = 735$ ways.
Specific behaviours
✓ indicates correct method   ✓ correct number of ways

(b) Given ${}^nP_r = 90$ and ${}^nC_r = 45$, determine the value of n and the value of r. (4 marks)

Solution

$$90 = \frac{n!}{(n-r)!} \quad (1)$$
$$45 = \frac{n!}{r!\,(n-r)!} \quad (2)$$

$$(1) \div (2) \Rightarrow \frac{90}{45} = r!$$
$$r! = 2 \Rightarrow r = 2$$

$$\text{Subtitute } r = 2 \text{ into } (1)$$
$$90 = \frac{n \times (n-1) \times (n-2)!}{(n-2)!}$$
$$n^2 - n - 90 = 0$$
$$(n+9)(n-10) = 0$$
$$\Rightarrow n = 10, -9$$

But $n \geq r$, hence $n = 10$.

Specific behaviours

✓ sets up two equations using permutation and combination formulae
✓ solves for r
✓ substitutes r and simplifies to quadratic equation
(CAS won't solve with the factorials)
✓ solves both values of n then justifies discarding $n = -9$

Question 13 **(9 marks)**

A sports club is selling raffle tickets. The tickets are left over from a previous event and are numbered consecutively from 462 to 900 inclusive.

(a) Determine how many of the tickets have a number that is

(i) a multiple of 6. (2 marks)

Solution
$\lfloor 900 \div 6 \rfloor - \lfloor 461 \div 6 \rfloor = 150 - 76 = 74$ tickets.
Specific behaviours
✓ multiples between 1 and 900 ✓ subtracts multiples < 462 to obtain correct number

(ii) a multiple of 6 or a multiple of 21. (3 marks)

Solution
Multiples of 21: $\lfloor 900 \div 21 \rfloor - \lfloor 461 \div 21 \rfloor = 42 - 21 = 21$ Multiples of 42 (i.e., multiples of both using LCM of 6 and 21): $\lfloor 900 \div 42 \rfloor - \lfloor 461 \div 42 \rfloor = 21 - 10 = 11$ Using the inclusion-exclusion principle: $n = 74 + 21 - 11 = 84$ tickets.
Specific behaviours
✓ multiples of 21 in range ✓ multiples of 42 in range ✓ correctly uses inclusion-exclusion principle to obtain correct number

(iii) a multiple of 6 or a multiple of 21 but not a multiple of both. (1 mark)

Solution
$n = 84 - 11 = 73$ tickets.
Specific behaviours
✓ correct number

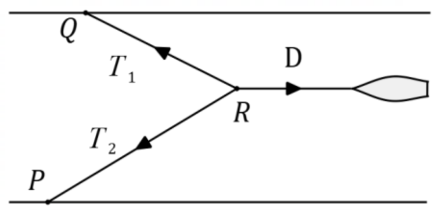
Once all these leftover tickets have been sold, they will be placed in a large bucket, thoroughly mixed and drawn from the bucket at random and without replacement.

(b) After how many draws can you be certain that at least four of the tickets taken from the bucket have numbers that start with the same digit? Justify your answer. (3 marks)

Solution
6 pigeon-holes (starting digits $4, 5, 6, 7, 8, 9$); however, there is only 1 ticket starting with 9. Worst case scenario is to select the ticket starting with 9 and 3 of each of $\{4,5,6,7,8\}$, before the final ticket selected will complete the set of 4: $1 + 5 \times 3 + 1 = 17$ pigeons. You can be certain after 17 draws.
Specific behaviours
✓ identifies the 6 pigeonholes ✓ determines correct number of draws ✓ justifies answer *If students answer 19 (forgetting there is only one 9) deduct 1 mark then follow through.*

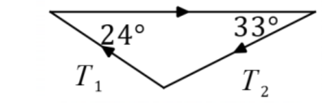
Question 14 **(8 marks)**

The diagram below shows a small boat is moored by ropes QR and PR that make angles of $24°$ and $33°$ respectively with the banks of the river.



(a) Determine T_1 and T_2, the tensions in each rope, when the force D, caused by the water current acting on the boat, is 320 N. (4 marks)

Solution



Using vector triangle:

$$\frac{T_1}{\sin 33°} = \frac{T_2}{\sin 24°} = \frac{320}{\sin 123°}$$

$$T_1 = 208 \text{ N}, \qquad T_2 = 155 \text{ N}$$

Specific behaviours

✓ forms vector triangle using tensions and drag
✓ indicates correct use of sine rule
✓ one correct tension (reasonable rounding)
✓ second correct tension (reasonable rounding)

Alternate Solution

Converting to component form and equating for equilibrium:

$$\begin{cases} T_1 \cos 24° + T_2 \cos 33° = 320 \\ T_1 \sin 24° - T_2 \sin 33° = 0 \end{cases}$$

Solving on CAS:

$$T_1 = 208 \text{ N}, \qquad T_2 = 155 \text{ N}$$

Specific behaviours

✓ forms equation using $\boldsymbol{i}$ component
✓ forms equation using $\boldsymbol{j}$ component
✓ one correct tension (reasonable rounding)
✓ second correct tension (reasonable rounding)

A steady current of 0.7 m/s flows eastwards, and the parallel banks of the river are 170 m apart.

(b) Another small boat leaves P at a speed of 4 m/s and heads slightly upstream, steering a course that makes an angle of $85°$ with the bank of the river. Determine how far the boat is from Q when it reaches the opposite bank, given that Q is 25 m downstream from P.

(4 marks)

Solution

Time to cross river: $170 \div (4 \sin 85°) = 42.7$ s.

Eastward speed of boat: $0.7 - 4 \cos 85° = 0.351$ m/s.

Eastward displacement: $42.7 \times 0.354 = 15.0$ m and so the boat will arrive at a point $25 - 15 = 10$ m upstream from Q.

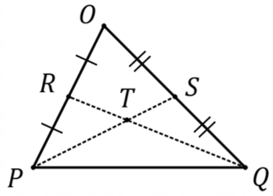
Specific behaviours

✓ indicates speed perpendicular to riverbank, i.e. $4 \sin 85°$
✓ obtains time to cross river
✓ indicates velocity parallel to riverbank
✓ correct distance from Q (no need to indicate upstream)

Question 15 **(8 marks)**

In ΔOPQ, R and S are the midpoints of sides OP and OQ respectively. PS intersects QR at T.

(a) Sketch a diagram to show this information. (1 mark)

Solution

Specific behaviours
✓ correctly labelled diagram

Let $\underset{\sim}{p} = \overrightarrow{OP}$, $\underset{\sim}{q} = \overrightarrow{OQ}$, $\overrightarrow{PT} = \lambda \times \overrightarrow{PS}$ and $\overrightarrow{QT} = \mu \times \overrightarrow{QR}$.

(b) Express $\overrightarrow{OT}$ in terms of $\underset{\sim}{p}, \underset{\sim}{q}$ and λ. (2 marks)

Solution
$\overrightarrow{OT} = \overrightarrow{OP} + \overrightarrow{PT}$ $= \overrightarrow{OP} + \lambda\overrightarrow{PS}$ $= \underset{\sim}{p} + \lambda\left(-\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q}\right)$ $= (1-\lambda)\underset{\sim}{p} + \frac{\lambda}{2}\underset{\sim}{q}$
Specific behaviours
✓ correct vector for $\overrightarrow{PS}$ ✓ correct expression (ok if 3rd line)

(c) Express $\overrightarrow{OT}$ in terms of $\underset{\sim}{p}, \underset{\sim}{q}$ and μ. (1 mark)

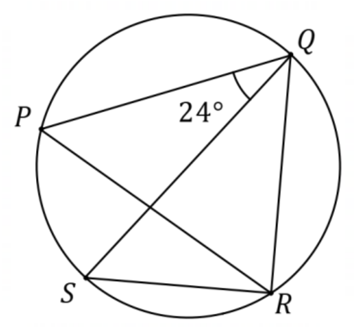
Solution
$\overrightarrow{OT} = \overrightarrow{OQ} + \overrightarrow{QT}$ $= \overrightarrow{OQ} + \lambda\overrightarrow{QR}$ $= \underset{\sim}{q} + \mu\left(-\underset{\sim}{q} + \frac{1}{2}\underset{\sim}{p}\right)$ $= \frac{\mu}{2}\underset{\sim}{p} + (1-\mu)\underset{\sim}{q}$
Specific behaviours
✓ correct expression (OK if 3rd line)

(d) Prove that $\overrightarrow{OP} + \overrightarrow{OQ} = 3 \times \overrightarrow{OT}$. (4 marks)

Solution
From parts (b) and (c) above we can equate both vectors for $\overrightarrow{OT}$: $\underset{\sim}{p} + \lambda\left(-\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q}\right) = \underset{\sim}{q} + \mu\left(-\underset{\sim}{q} + \frac{1}{2}\underset{\sim}{p}\right)$ Considering $\underset{\sim}{p}$ and $\underset{\sim}{q}$ coefficients: $1 - \lambda = \frac{\mu}{2}, \quad \frac{\lambda}{2} = 1 - \mu$ Solving simultaneously, $\lambda = \mu = \frac{2}{3}$. Hence $3\overrightarrow{OT} = 3\left(\underset{\sim}{p} + \frac{2}{3}\left(-\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q}\right)\right) = 3\underset{\sim}{p} - 2\underset{\sim}{p} + \underset{\sim}{q} = \underset{\sim}{p} + \underset{\sim}{q} = \overrightarrow{OP} + \overrightarrow{OQ}$
Specific behaviours
✓ equates vectors for $\overrightarrow{OT}$ ✓ forms equations using coefficients ✓ solves for at least one unknown ✓ completes proof

Question 16 **(6 marks)**

(a) In the diagram, points P, Q, R and S lie on a circle so that QS is a diameter and $PQ = PR$. Determine the size of $\angle RQS$ when $\angle PQS = 24°$.
Reasons are not required. (3 marks)



Solution

$\angle PRS = \angle PQS = 24°$ (Angles on same arc)

$\angle PRQ = 90° - 24° = 66°$ (Angle in semicircle)

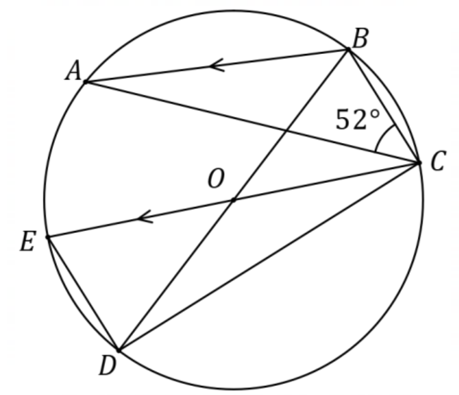
ΔPQR is isosceles, so $\angle PQR = \angle PRQ$.

$$\begin{aligned}\angle RQS &= \angle PQR - \angle PQS \\ &= 66° - 24° \\ &= 42°\end{aligned}$$

Specific behaviours

✓ obtains $\angle PRS$
✓ obtains $\angle PRQ$
✓ correct $\angle RQS$

(b) In the diagram, points A, B, C, D and E lie on a circle with centre O. BD and CE are diameters, and AB is parallel to EC. Determine the size of $\angle OED$ when $\angle BCA = 52°$.
(3 marks)



Solution

$\angle BCD = 90°$ (angle in semicircle)
$\Rightarrow \angle ACD = 90° - 52° = 38°$
Hence, $\angle ABD = 38°$ ($\angle$'s in same segment equal)

acute $\angle EOD = \angle ABD = 38°$ (corresponding $\angle$'s of parallel lines equal)

Since $OE = OD$ (radii), then ΔOED is isosceles

$\Rightarrow \angle OED = \angle ODE$ (base angles equal)
$= \frac{180° - 38°}{2} = 71°$ (triangle angle sum)

Specific behaviours

✓ obtains $\angle ABD$ with reasons
✓ obtains $\angle EOD$ with reasons
✓ correct $\angle OED$ with reasons

Alternate Solution

Angle in semicircle and angles on same arc:
$$\angle ABD = \angle ACD = 90° - 52° = 38°$$

Let $\angle OED = \angle OBC = \angle OCB = x$.

Sum of co-interior angles:
$$\begin{aligned}\angle ABC + \angle BCO &= 180° \\ 38° + x + x &= 180° \\ x = \angle OED &= 71°\end{aligned}$$

Specific behaviours

✓ obtains $\angle ABD$ with reasons
✓ forms equation using co-interior angles
✓ correct $\angle OED$ with reasons

Question 17 **(7 marks)**

Five cubes, identical apart from their colour, are to be placed one on top of another to form a tower. Determine the number of different towers that can be made using:

(a) eight different coloured cubes. (1 mark)

Solution
$^8P_5 = 6720$
Specific behaviours
✓ correct number of towers

(b) a blue, a green and three red cubes. (1 mark)

Solution
$\frac{5!}{3!} = 20$
Specific behaviours
✓ correct number of towers

(c) three green and two red cubes. (1 mark)

Solution
$\frac{5!}{3!\,2!} = 10$
Specific behaviours
✓ correct number of towers

(d) one green, one blue, one yellow, one pink, two white and two red cubes. (4 marks)

Solution

A. All different coloured cubes, arrange.

$$n_A = {}^6P_5 = 720$$

B. Choose 2 same colour, choose 3 different, arrange.

$$n_B = {}^2C_1 \times {}^5C_3 \times \frac{5!}{2!} = 2 \times 10 \times 60 = 1200$$

C. Choose 2 same colour pairs, choose 1 different, arrange.

$$n_C = {}^2C_2 \times {}^4C_1 \times \frac{5!}{2!\,2!} = 1 \times 4 \times 30 = 120$$

Total of $720 + 1200 + 120 = 2040$ different towers.

Specific behaviours

✓ correctly splits into mutually exclusive and exhaustive cases
✓ one correct number of arrangements
✓ second correct number of arrangements
✓ correct total number of arrangements

Question 18 **(6 marks)**

Use an algebraic method to prove the following conjecture is true:

"For every $k \in \mathbb{Z}$, $k^2 - 8k + 4$ is odd if and only if k is odd."

Solution
'Backwards' proof: RTP: If k is odd, then $k^2 - 8k + 4$ is odd. Let $k = 2n + 1,\ n \in \mathbb{Z}$ $k^2 - 8k + 4 = (2n+1)^2 - 8(2n+1) + 4$ $= 4n^2 + 4n + 1 - 16n - 8 + 4$ $= 4n^2 - 12n - 3$ $= 2(2n^2 - 6n - 2) + 1\ or\ 2(2n^2 - 6n - 1) - 1$ Hence, $k^2 - 8k + 4$ is odd. Therefore, this is true. 'Forwards' proof: RTP: If $k^2 - 8k + 4$ is odd, then k is odd. Instead, prove the contrapositive, i.e. RTP: If k is even, then $k^2 - 8k + 4$ is even. Let $k = 2n,\ n \in \mathbb{Z}$ $k^2 - 8k + 4 = (2n)^2 - 8(2n) + 4$ $= 4n^2 - 16n + 4$ $= 2(2n^2 - 8n + 2)$ Hence, $k^2 - 8k + 4$ is even. Therefore, the contrapositive is true and so is the original. Hence, since the statement is true in both directions, then the conjecture is true. i.e. For every $k \in \mathbb{Z}$, $k^2 - 8k + 4$ is odd if and only if k is odd.

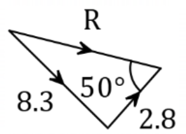
Specific behaviours
✓ (for backwards) defines k and n, and substitutes into expression ✓ (for backwards) correctly rewrites to an 'odd' form and concludes ✓ (for forwards) rewrites with the contrapositive ✓ (for forwards) defines k and n, and substitutes into expression ✓ (for forwards) correctly rewrites to an 'even' form and concludes ✓ makes final conclusion referring to both results

Question 19 **(9 marks)**

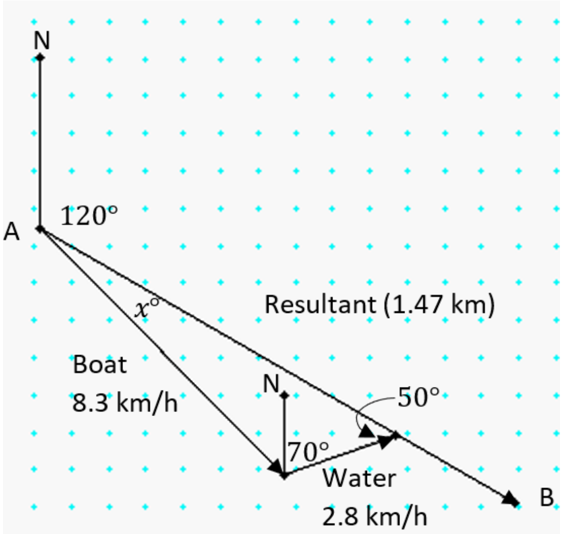
Water in a large river estuary flows at a constant 2.8 km/h on a bearing $070°T$. At $1{:}45$ pm, a fishing boat in the estuary has just dropped a crab pot at A and heads off towards B to drop its last one, 1.47 km away on a bearing of $120°T$. The fishing boat travels at a constant 8.3 km/h.

(a) Sketch a neat, clearly labelled diagram to represent the situation, then determine the bearing the boat should steer from A to B. (4 marks)

Solution



Or



Angle between resultant and water flow:

$$120° - 70° = 50°$$

$$\frac{\sin x}{2.8} = \frac{\sin 50°}{8.3} \Rightarrow x = 14.98°, 165.02°$$

But $x < 50°$ since $2.8 < 8.3$.

Hence, bearing to steer is

$$120° + 15° = 135°T$$

Specific behaviours

✓ correct angle in triangle (i.e. $50°$)
✓ correctly shows sum of velocity vectors and clearly labelled
✓ indicates equation for angle
✓ correct angle and bearing

Once the boat reaches B it will spend 5 minutes dropping the last pot and then head off to an anchorage 2.2 km away from B on a bearing of $250°T$.

(b) Determine the time at which the boat is expected to reach the anchorage. (5 marks)

Solution
Let magnitude of resultant be v km/h and $180° - 50° - 15° = 115°$: $$\frac{v}{\sin 115°} = \frac{8.3}{\sin 50°} \Rightarrow v = 9.82 \text{ km/h}$$ Or, using cosine rule: $$8.3^2 = v^2 + 2.8^2 - 2 \times v \times 2.8 \times \cos 50°$$ $$CAS \Rightarrow v = 9.82\ km/h$$ Time from A to B is $1.47 \div 9.82 \times 60 = 9$ minutes.   On way to anchorage, $250° - 070° = 180°$, so boat heading directly against current, and its resultant velocity is $8.3 - 2.8 = 5.5$ km/h.   Time from B to anchorage is $2.2 \div 5.5 \times 60 = 24$ minutes   Total time taken = $9 + 5 + 24 = 38$ minutes   So arrival time: $1{:}45 \text{ pm} + 38 \text{ min} = 2{:}23$ pm.
Specific behaviours
✓ indicates equation for magnitude of resultant   ✓ obtains magnitude of resultant   ✓ correct time from A to B ✓ indicates correct resultant velocity from B to anchorage   ✓ correct arrival time at anchorage

End of questions

Question 19 **ALTERNATE METHOD** **(9 marks)**

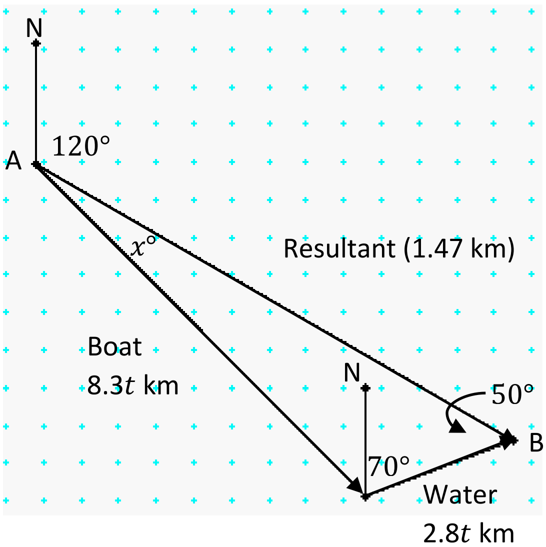
Water in a large river estuary flows at a constant 2.8 km/h on a bearing $070°T$. At $1{:}45$ pm, a fishing boat in the estuary has just dropped a crab pot at A and heads off towards B to drop its last one, 1.47 km away on a bearing of $120°T$. The fishing boat travels at a constant 8.3 km/h.

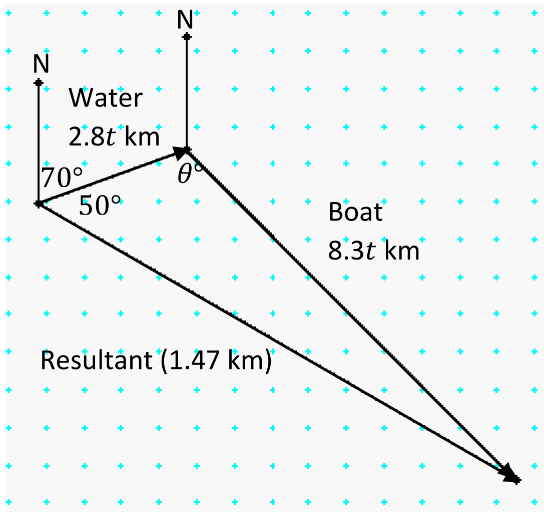
(a) Sketch a neat, clearly labelled diagram to represent the situation, then determine the bearing the boat should steer from A to B. (4 marks)

Alternate Solution

Let the journey from A to B take t hours.

Diagram (either version acceptable):





Angle between resultant and water flow:

$$120° - 70° = 50°$$

Using the cosine rule:

$$(8.3t)^2 = 1.47^2 + (2.8t)^2 - 2 \times 1.47 \times 2.8t \times \cos 50°$$

$$CAS \Rightarrow t = 0.1497, \cancel{-0.2364}\ (t > 0)$$

Hence, the distance along the "boat" vector is $8.3 \times 0.1497 = 1.24\ km$
and the distance along the "water" vector is $2.8 \times 0.1497 = 0.42\ km$

For the first diagram, applying the cosine rule to solve $x°$ gives

$$0.42^2 = 1.47^2 + 1.24^2 - 2 \times 1.47 \times 1.24 \times \cos x° \qquad (*)$$

$$\Rightarrow x = 14.98°$$

Hence, bearing to steer is

$$120° + 15° = 135°T$$

For the second diagram, applying the cosine rule to solve $\theta°$ gives

$$1.47^2 = 0.42^2 + 1.24^2 - 2 \times 0.42 \times 1.24 \times \cos \theta°$$

$$\Rightarrow \theta = 115°$$

Hence, bearing to steer is

$$270° - (115° + 20°) = 135°T$$

Specific behaviours
✓ correct angle in triangle (i.e. $50°$)
✓ correctly shows sum of displacement vectors and clearly labelled
✓ indicates equation for angle, i.e. the (*) line or sine rule to solve angle
✓ correct angle and bearing

Once the boat reaches B it will spend 5 minutes dropping the last pot and then head off to an anchorage 2.2 km away from B on a bearing of $250°T$.

(b) Determine the time at which the boat is expected to reach the anchorage. (5 marks)

Alternate Solution
As before, using the cosine rule: $$(8.3t)^2 = 1.47^2 + (2.8t)^2 - 2 \times 1.47 \times 2.8t \times \cos 50°$$ $$CAS \Rightarrow t = 0.1497, \cancel{-0.2364}\ (t > 0)$$ Time from A to B is $0.1497 \times 60 = 9$ minutes. On way to anchorage, $250° - 070° = 180°$, so boat heading directly against current, and its resultant velocity is $8.3 - 2.8 = 5.5$ km/h. Time from B to anchorage is $2.2 \div 5.5 \times 60 = 24$ minutes Total time taken = $9 + 5 + 24 = 38$ minutes So arrival time: $1{:}45\ \text{pm} + 38\ \text{min} = 2{:}23$ pm.
Specific behaviours
✓ attempts to use the cosine rule to determine equation for t ✓ correct substitution into cosine rule (including brackets) *Allow the above two marks if this was already done in (a)* ✓ correct time from A to B ✓ indicates correct resultant velocity from B to anchorage ✓ correct arrival time at anchorage