Rossmoyne Senior High School

Semester One Examination, 2023

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Number of additional answer booklets used (if applicable):

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	91	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (50 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

OVERALL FOR THE EXAM, FOLLOW THE BEHAVIOURS FOR 2 MARK QUESTIONS

Question 1 **(6 marks)**

Relative to an origin O, the position vectors of points K, L and M are $\begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}-1\\-7\end{pmatrix}$ and $\begin{pmatrix}-3\\-1\end{pmatrix}$ respectively.

(a) Determine $\overrightarrow{LK}$. (1 mark)

Solution
$\overrightarrow{LK} = \begin{pmatrix}5\\1\end{pmatrix} - \begin{pmatrix}-1\\-7\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix}$
Specific behaviours
✓ correct vector

(b) Determine the magnitude of $\overrightarrow{LK}$. (1 mark)

Solution
$\left\lvert\overrightarrow{LK}\right\rvert = \sqrt{36+64} = 10$
Specific behaviours
✓ correct magnitude

(c) Determine a vector in the same direction as $\overrightarrow{LK}$ that has the same magnitude as $\overrightarrow{LM}$. Simplify your answer.

(4 marks)

Solution
$\overrightarrow{LM} = \begin{pmatrix}-3\\-1\end{pmatrix} - \begin{pmatrix}-1\\-7\end{pmatrix} = \begin{pmatrix}-2\\6\end{pmatrix}$ $\left\lvert\overrightarrow{LM}\right\rvert = \sqrt{4+36} = 2\sqrt{10}$ $Unit\ vec\ \overrightarrow{LK} = \frac{1}{10}\begin{pmatrix}6\\8\end{pmatrix}$ $\underset{\sim}{r} = 2\sqrt{10} \times \frac{1}{10}\begin{pmatrix}6\\8\end{pmatrix}$ $= \frac{\sqrt{10}}{5}\begin{pmatrix}6\\8\end{pmatrix}$
Specific behaviours
✓ determines $\overrightarrow{LM}$ ✓ determines magnitude of $\overrightarrow{LM}$ ✓ determines unit vector $\overrightarrow{LK}$ (ok if within final calculation) follow through, if needed, from student's (b)   ✓ correct final vector

Question 2 **(6 marks)**

A student, mindful of the fact that a square, rectangle and rhombus are special cases of a parallelogram, wrote the following true statement:

If a quadrilateral is a parallelogram, then opposite angles in the quadrilateral are congruent.

(a) Write the converse of the statement and state whether it is true. (2 marks)

Solution
If opposite angles in the quadrilateral are congruent, then the quadrilateral is a parallelogram. Converse statement is true.
Specific behaviours
✓ correct converse statement ✓ states true

(b) Write the inverse of the statement and state whether it is true. (2 marks)

Solution
If the quadrilateral is not a parallelogram, then opposite angles in the quadrilateral are not congruent. Inverse statement is true.
Specific behaviours
✓ correct inverse statement ✓ states true

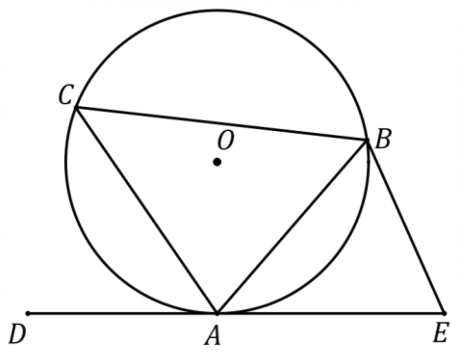
(c) Write the contrapositive of the statement and state whether it is true. (2 marks)

Solution
If opposite angles in the quadrilateral are not congruent, then the quadrilateral is not a parallelogram. Contrapositive statement is true.
Specific behaviours
✓ correct contrapositive statement ✓ states true

Question 3 **(7 marks)**

(a) In the diagram, DE is tangential to the circle, with centre O, at A and the vertices of triangle ABC lie on the circle as shown.

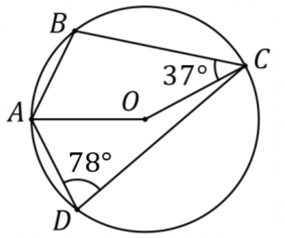
Prove the "angle in the alternate segment theorem", i.e. that $\angle BAE = \angle BCA$.



(4 marks)

Solution
Let $\angle BAE = x$. $\angle BAO = 90° - x$ (tangent DE perp. to radius OA) ΔOAB is isosceles as OA, OB are radii $\angle BOA = 180° - 2(90° - x) = 2x$ (base angles equal and triangle sum $180°$) $\angle BCA = \frac{1}{2}\angle BOA = \frac{1}{2}(2x) = x$ (central angle theorem) Hence $\angle BAE = \angle BCA = x$ as required. *Note that several alternative proofs exist.*
Specific behaviours
✓ expression for $\angle BAO$ ✓ expression for $\angle BOA$ ✓ expression for $\angle BCA$ ✓ completes proof and supplies adequate explanation throughout

(b) In the diagram below, A, B, C and D lie on circle with centre O. Determine the size of $\angle OAB$ when $\angle OCB = 37°$ and $\angle ADC = 78°$. Reasons are not required. (3 marks)



Solution
$\angle AOC = 2 \times 78° = 156°$ $\angle ABC = 180° - 78° = 102°$ $\angle OAB = 360° - 37° - 156° - 102° = 65°$
Specific behaviours
✓ correctly obtains angle at O ✓ correctly obtains angle at B ✓ correct angle

Question 4 **(9 marks)**

Consider the vectors $\underset{\sim}{r} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$, $\underset{\sim}{s} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underset{\sim}{t} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$. Determine

(a) a unit vector in the same direction as $\underset{\sim}{r} + 8\underset{\sim}{s}$. (2 marks)

Solution
$\underset{\sim}{r} + 8\underset{\sim}{s} = \begin{pmatrix} 5 \\ -8 \end{pmatrix} + 8\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 21 \\ 0 \end{pmatrix} \Rightarrow \underset{\sim}{\hat{u}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Specific behaviours
✓ correct sum ✓ correct unit vector

(b) vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ given that $\underset{\sim}{r} = \underset{\sim}{a} - 2\underset{\sim}{b}$ and $\underset{\sim}{s} = \underset{\sim}{a} + \underset{\sim}{b}$. (3 marks)

Solution
Subtracting $\underset{\sim}{r}$ from $\underset{\sim}{s}$ gives $\underset{\sim}{s} - \underset{\sim}{r} = 3\underset{\sim}{b}$ and so $\underset{\sim}{b} = \frac{1}{3}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ -8 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ $\underset{\sim}{a} = \underset{\sim}{r} + 2\underset{\sim}{b} = \begin{pmatrix} 5 \\ -8 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$
Specific behaviours
✓ uses equations to eliminate one unknown vector ✓ obtains one vector ✓ obtains second vector

(c) the value of the constants λ and μ when $\lambda\underset{\sim}{r} + \mu\underset{\sim}{s} = \underset{\sim}{t}$. (4 marks)

Solution
$\lambda\begin{pmatrix} 5 \\ -8 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix} \Rightarrow \begin{cases} 5\lambda + 2\mu = 5 \\ -8\lambda + \mu = 13 \end{cases}$ $5\lambda + 2\mu = 5$ $16\lambda - 2\mu = -26$ $21\lambda = -21$ $\lambda = -1$ $\mu = 13 + 8(-1) = 5$ Hence $\lambda = -1, \mu = 5$.
Specific behaviours
✓ equation using $\underset{\sim}{i}$ coefficients ✓ equation using $\underset{\sim}{j}$ coefficients ✓ solves equation for one value ✓ correct values

Question 5 **(7 marks)**

(a) The letters of the word MATAMATA are arranged randomly in a line. Determine

(i) the number of different ways this can be done. (2 marks)

Solution
8 letters, $4\times$A, $2\times$M, $2\times$T. Number of arrangements is: $$\frac{8!}{4!\,2!\,2!}=\frac{8\times7\times6\times5}{2\times2}=10\times42=420$$
Specific behaviours
✓ correct expression using factorials ✓ correct number of arrangements

(ii) the number of arrangements in which all the A's are grouped together. (2 marks)

Solution
Let AAAA be one object (with one arrangement). Then arrange this object, $2\times$M, $2\times$T. Number of arrangements is: $$\frac{5!}{2!\,2!}=\frac{5\times4\times3\times2}{2\times2}=30$$
Specific behaviours
✓ correct expression using factorials ✓ correct number of arrangements

(b) A bag consists of 10 blocks, numbered 0 to 9 inclusive.
Show that if someone selects 7 blocks at random, then there are two blocks which will sum to 10. (3 marks)

Solution
Pairing the blocks to make sums of 10 gives: $$(1,9),(2,8),(3,7),(4,6)$$ Plus the blocks 0 and 5 which do not have a matching pair. The pairs, 0 and 5 form the 6 pigeonholes. Hence, by the Pigeonhole Principle, selecting 7 blocks will ensure there is a full pair selected, i.e. there are two block which will sum to 10.
Specific behaviours
✓ identifies pairs of blocks ✓ indicates the 0 and 5 are included in counting the number of pigeonholes ✓ uses Pigeonhole Principle to justify conclusion (must state "Pigeonhole Principle" for this mark)

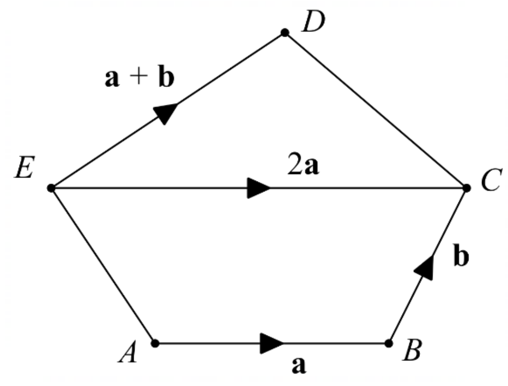
Question 6 **(8 marks)**

(a) Consider parallelogram $OKLM$, so that $\overrightarrow{OK} = \underset{\sim}{k}$ and $\overrightarrow{OM} = \underset{\sim}{m}$. Use a vector method to show that when diagonals $\overrightarrow{OL}$ and $\overrightarrow{KM}$ are perpendicular, then the parallelogram is a rhombus.

(4 marks)

Solution
$\overrightarrow{OL} = \underset{\sim}{k} + \underset{\sim}{m}, \qquad \overrightarrow{KM} = \underset{\sim}{m} - \underset{\sim}{k}$ Vectors are perpendicular: $\left(\underset{\sim}{k} + \underset{\sim}{m}\right) \cdot \left(\underset{\sim}{m} - \underset{\sim}{k}\right) = 0$ $\underset{\sim}{k} \cdot \underset{\sim}{m} + \underset{\sim}{m} \cdot \underset{\sim}{m} - \underset{\sim}{k} \cdot \underset{\sim}{k} - \underset{\sim}{m} \cdot \underset{\sim}{k} = 0$ $\underset{\sim}{m} \cdot \underset{\sim}{m} - \underset{\sim}{k} \cdot \underset{\sim}{k} = 0$ $\left\|\underset{\sim}{m}\right\|^2 = \left\|\underset{\sim}{k}\right\|^2$ Since $\left\|\underset{\sim}{m}\right\| \geq 0, \left\|\underset{\sim}{k}\right\| \geq 0$, then $\left\|\underset{\sim}{m}\right\| = \left\|\underset{\sim}{k}\right\|$. Hence lengths of all sides of $OKLM$ are equal and it is a rhombus.
Specific behaviours
✓ states vectors for both diagonals ✓ uses perpendicularity to form scalar product equation ✓ expands and simplifies scalar product to get $\left\|\underset{\sim}{m}\right\|^2 = \left\|\underset{\sim}{k}\right\|^2$ with correct vector notation (i.e. underline, |…| magnitude etc.) used throughout ✓ refers to $\left\|\underset{\sim}{m}\right\|, \left\|\underset{\sim}{k}\right\| \geq 0$ to conclude equal magnitudes and deduce shape is a rhombus

(b) $ABCDE$ is a pentagon, with $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{EC} = 2\mathbf{a}$ and $\overrightarrow{ED} = \mathbf{a} + \mathbf{b}$.



(i) Determine an expression for $\overrightarrow{CD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (1 mark)

Solution
$\overrightarrow{CD} = \overrightarrow{CE} + \overrightarrow{ED}$ $\overrightarrow{CD} = -2\mathbf{a} + \mathbf{a} + \mathbf{b}$ $\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$
Specific behaviours
✓ Determines simplified expression for $\overrightarrow{CD}$

(ii) Determine an expression for $\overrightarrow{AE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (1 mark)

Solution
$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CE}$ $\overrightarrow{AE} = \mathbf{a} + \mathbf{b} - 2\mathbf{a}$ $\overrightarrow{AE} = \mathbf{b} - \mathbf{a}$
Specific behaviours
✓ Determines simplified expression for $\overrightarrow{AE}$

(iii) Hence, state **two** geometric facts about the sides CD and AE. (2 marks)

Solution
Since $\overrightarrow{CD} = \overrightarrow{AE}$, CD and AE are parallel and of equal length
Specific behaviours
✓ States the sides are parallel ✓ States the sides are the same length

Question 7 **(7 marks)**

(a) Vectors $\underset{\sim}{m}$ and $\underset{\sim}{n}$ have magnitudes 3 and 5 respectively, and $\underset{\sim}{m} \cdot \underset{\sim}{n} = -4$. Evaluate

(i) $\left(6\underset{\sim}{m}\right) \cdot \left(\frac{1}{3}\underset{\sim}{n}\right)$. (1 mark)

Solution
$\left(6\underset{\sim}{m}\right) \cdot \left(\frac{1}{3}\underset{\sim}{n}\right) = 2\left(\underset{\sim}{m} \cdot \underset{\sim}{n}\right) = 2(-4) = -8$
Specific behaviours
✓ correct value

(ii) $\left(2\underset{\sim}{m} - \underset{\sim}{n}\right) \cdot \left(\underset{\sim}{m} - 3\underset{\sim}{n}\right)$. (3 marks)

Solution
$\left(2\underset{\sim}{m} - \underset{\sim}{n}\right) \cdot \left(\underset{\sim}{m} - 3\underset{\sim}{n}\right) = 2\underset{\sim}{m} \cdot \underset{\sim}{m} - \underset{\sim}{n} \cdot \underset{\sim}{m} - 6\underset{\sim}{m} \cdot \underset{\sim}{n} + 3\underset{\sim}{n} \cdot \underset{\sim}{n}$ $= 2\left\lvert\underset{\sim}{m}\right\rvert^2 + 3\left\lvert\underset{\sim}{n}\right\rvert^2 - 7\underset{\sim}{m} \cdot \underset{\sim}{n}$ $= 2(3)^2 + 3(5)^2 - 7(-4)$ $= 18 + 75 + 28 = 121$
Specific behaviours
✓ correctly expands using scalar products   ✓ simplifies product using $\underset{\sim}{r} \cdot \underset{\sim}{r} = \left\lvert\underset{\sim}{r}\right\rvert^2$ (OK if this is done in the first step)   ✓ correct value

(b) The vector projection of $\begin{pmatrix} 5 \\ m \end{pmatrix}$ on $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ is $\begin{pmatrix} 6 \\ -6 \end{pmatrix}$. Determine the value of the constant m. (3 marks)

Solution
Vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is $\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b}$ or $\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\left\lvert\underset{\sim}{b}\right\rvert^2}\right)\underset{\sim}{b}$ or $\left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$, and so $\frac{\begin{pmatrix} 5 \\ m \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}}{\begin{pmatrix} -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$ $\frac{\begin{pmatrix} 5 \\ m \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}}{\begin{pmatrix} -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix}} = -3$ $\frac{2m - 10}{8} = -3 \quad (*)$ $2m - 10 = -24$ $m = -7$
Specific behaviours
✓ forms equation using vector projection   ✓ correctly evaluates scalar products to get an equation not involving vectors   *equivalent to the (*) equation, but may be from matching $\boldsymbol{i}$ or $\boldsymbol{j}$ components*   ✓ correct value of m

End of questions

Supplementary page

Question number: _________