Rossmoyne Senior High School

Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures

In words ______________________

Your name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	94	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

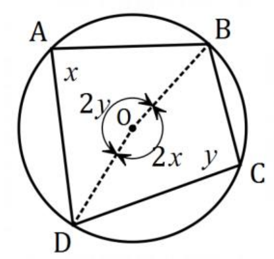
Section Two: Calculator-assumed **65% (94 Marks)**

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(3 marks)**

Prove that the opposite angles of a cyclic quadrilateral are supplementary.

Solution

Required to prove that $x + y = 180°$ $B\hat{O}D = 2 \times B\hat{A}D = 2x$ (Angle at centre-circumference) $B\check{O}D = 2 \times B\hat{C}D = 2y$ (Angle at centre-circumference) $2x + 2y = 360°$ (Angle at a point) Hence $x + y = 180°$ as required.
Specific behaviours
✓ diagram with labels or angle markings to support working ✓ uses angles at centre and on circumference ✓ completes proof (only one pair needs to be proved)

Question 10 **(6 marks)**

(a) Triangle ABC has vertices $A(2,-3), B(2,5)$ and $C(12,-1)$. Determine the area of this triangle after it has been transformed using the matrix $\begin{bmatrix} -4 & 4 \\ 3 & 3 \end{bmatrix}$. (3 marks)

Solution
Area of $\Delta ABC = \frac{1}{2} \times 8 \times 10 = 40$. Determinant of transformation matrix $= -24$. Area of transformed triangle $= \lvert -24 \rvert \times 40 = 960$ square units.
Specific behaviours
✓ area of ΔABC ✓ correct use of determinant to identify factor increase ✓ correct area

(b) Show use of matrix algebra, including any inverse matrix used, to solve the following system of linear equations: (3 marks)

$$2a + 3b = 55$$
$$4a + 5b = 79$$

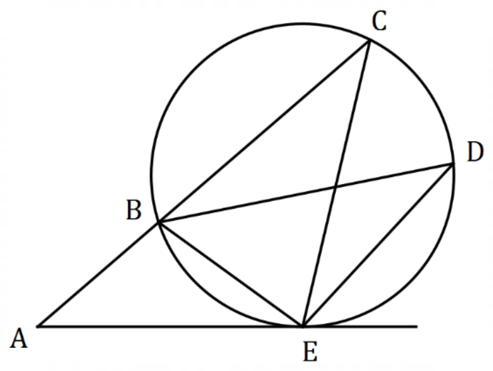
Solution
$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 55 \\ 79 \end{bmatrix}$ $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 55 \\ 79 \end{bmatrix}$ $= \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 55 \\ 79 \end{bmatrix} \quad \left(\text{or uses } \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}\right)$ $= \begin{bmatrix} -19 \\ 31 \end{bmatrix}$ $\therefore a = -19, b = 31$
Specific behaviours
✓ writes system in matrix form ✓ matrix expression for solution, including inverse ✓ correct solution *correct answer without matrix algebra = 1 mark only

Question 11 **(7 marks)**

(a) In the diagram shown (not to scale) ABC is a straight line and B, C, D and E lie on a circle.

AE is a tangent to the circle at E, $\angle BEC = 76°$ and $\angle BDE = 27°$.

Determine, with reasons, the size of $\angle BAE$.



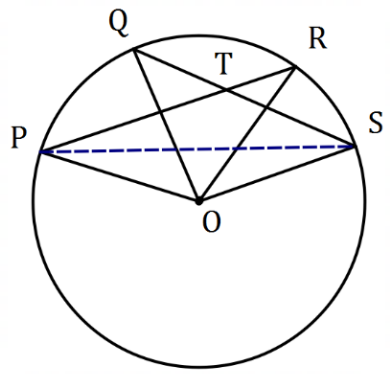
(4 marks)

Solution
$\angle BCE = 27°$ (angles on same arc BE) $\angle BEA = 27°$ (alternate segment theorem) $\angle AEC = 76° + 27° = 103°$ (adjacent angles) $\angle BAE = 180° - 103° - 27° = 50°$ (angle sum in ΔAEC)
Specific behaviours
✓ $\angle BCE$ with reason ✓ $\angle BEA$ with reason ✓ $\angle AEC$ with reason ✓ $\angle BAE$ with reason

(b) In the diagram shown (not to scale) P, Q, R and S lie on a circle centre O and chords QS and PR intersect at T.

$\angle POQ = 42°$ and $\angle ROS = 35°$.

Determine, with reasons, the size of $\angle RTS$.



(3 marks)

Solution
$\angle PSQ = \frac{1}{2} \times 42° = 21°$ (angle at centre-circumference) $\angle RPS = \frac{1}{2} \times 35° = 17.5°$ (angle at centre-circumference) $\angle RTS = 21° + 17.5° = 38.5°$ (sum of opposite interior angles)
Specific behaviours
✓ $\angle PSQ$ with reason ✓ $\angle RPS$ with reason ✓ $\angle RTS$ with reason

Question 12 **(8 marks)**

The vertices of triangle T are $A(-4,-3)$, $B(13,6)$ and $C(-2,8)$.

Transformation M is a translation by vector $\begin{bmatrix} 7 \\ -6 \end{bmatrix}$.

(a) State the coordinates of the image of A after triangle T is transformed by M. (1 mark)

Solution
$\begin{bmatrix} -4 \\ -3 \end{bmatrix} + \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \Rightarrow A'(3,-9)$
Specific behaviours
✓ correct coordinates Deduct mark if left as a matrix

Transformation N is a reflection in the line $y = x$.

(b) Determine the transformation matrix for N and state the coordinates of the image of B after triangle T is transformed by M and then by N. (3 marks)

Solution
$N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B' = \begin{bmatrix} 13 \\ 6 \end{bmatrix} + \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix}$ $B'' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$ $B''(0,20)$
Specific behaviours
✓ matrix for N ✓ transforms B by M ✓ coordinates of B'' (allow mark if left as a matrix)

Transformation P is a rotation of $45°$ clockwise about the origin.

(c) Determine the exact coordinates of the image of C after triangle T is transformed by N and then by P. (3 marks)

Solution
$P = \begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ $C' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ $C'' = P \times N \times \begin{bmatrix} -2 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} \\ -5\sqrt{2} \end{bmatrix}$ $C''(3\sqrt{2}, -5\sqrt{2})$
Specific behaviours
✓ matrix for P ✓ transforms C by N, or sets out the step $C'' = P \times N \times \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ ✓ coordinates of C'' (allow mark if left as a matrix)

(d) Write a matrix expression for the transformation matrix Q that represents the inverse of transformation N followed by the inverse of transformation P.
Leave your answer in terms of N and P. There is no need to simplify your expression. (1 mark)

Solution
N.B. N^{-1} can be replaced with N below, as N is self inverse. $Q = P^{-1} \times N^{-1}$ **Or** $Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$ **Or** $Q = (NP)^{-1}$ **Or** $Q = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\right)^{-1}$
Specific behaviours
✓ accept any correct expression

Question 13 **(8 marks)**

Two vectors are $\mathbf{p} = \begin{pmatrix} 72 \\ -154 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -39 \\ 252 \end{pmatrix}$. Determine

(a) the magnitude of $\mathbf{p}$. (1 mark)

Solution
$\sqrt{72^2 + 154^2} = 170$
Specific behaviours
✓ correct magnitude

(b) the angle between the directions of $\mathbf{q}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (2 marks)

Solution
$\cos^{-1}\dfrac{252}{255} = 8.8° = 0.154\, rad$ (or using CAS)
Specific behaviours
✓ indicates any correct method (e.g. dot product or right-angled triangle) ✓ correct angle at least 1 d.p. for degrees or 2 d.p. for radians

(c) the value of the scalar constant k so that $18\mathbf{p} + k\mathbf{q}$ is parallel to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (2 marks)

Solution
$18\begin{pmatrix} 72 \\ -154 \end{pmatrix} + k\begin{pmatrix} -39 \\ 252 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $18(-154) + 252k = 0$ $k = 11$
Specific behaviours
✓ equation with k ✓ value of k

(d) a vector $\mathbf{r}$ that is perpendicular to $\mathbf{p}$ with the magnitude of $\mathbf{q}$. (3 marks)

Solution
$\mathbf{r} = a\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 72 \\ -154 \end{pmatrix} = a\begin{pmatrix} 154 \\ 72 \end{pmatrix}$ $\lvert a \rvert = \dfrac{255}{170} \Rightarrow a = \pm 1.5$ $\mathbf{r} = \begin{pmatrix} 231 \\ 108 \end{pmatrix} \left(\text{or } \boldsymbol{r} = \begin{pmatrix} -231 \\ -108 \end{pmatrix} \right)$
Specific behaviours
✓ rotates vector 90° ✓ ratio of magnitudes ✓ any correct vector

Alternate Solution
Let $\boldsymbol{r} = \begin{pmatrix} a \\ b \end{pmatrix}$ $\Rightarrow \sqrt{a^2 + b^2} = \sqrt{39^2 + 252^2} = 255 \;\; (1)$ $\begin{pmatrix} a \\ b \end{pmatrix} . \begin{pmatrix} 72 \\ -154 \end{pmatrix} = 0$ $\Rightarrow 72a - 154b = 0 \;\; (2)$ Solving equations (1) and (2) $a = 231, b = 108 \;\; or \;\; a = -231, b = -108$ i.e. $\mathbf{r} = \begin{pmatrix} 231 \\ 108 \end{pmatrix} \left(\text{or } \boldsymbol{r} = \begin{pmatrix} -231 \\ -108 \end{pmatrix} \right)$
Specific behaviours
✓ determines equation using $\lvert \boldsymbol{q} \rvert$ ✓ determines equation using $\boldsymbol{r}.\boldsymbol{p} = 0$ ✓ solves for either correct vector

Question 14 **(8 marks)**

(a) Determine the number of integers between 1 and 749 that are

(i) divisible by 72. (1 mark)

Solution
$\lfloor 749 \div 72 \rfloor = 10$
Specific behaviours
✓ correct number

(ii) divisible by 8 or by 9 but not by 72. (3 marks)

Solution
Divisible by $8, 9$: $$\lfloor 749 \div 8 \rfloor = 93$$ $$\lfloor 749 \div 9 \rfloor = 83$$ Divisible by 8 or 9: $$93 + 83 - 10 = 166$$ Divisible by 8 or 9 but not 72: $$166 - 10 = 156$$
Specific behaviours
✓ numbers divisible by $8, 9$ ✓ number divisible by 8 or 9 ✓ correct answer

(b) A playlist offered by a music streaming service has 15 different songs. Every time a playlist is streamed, the songs are shuffled into a random arrangement.

Show that after the playlist has been streamed $75\,000$ times, at least 3 of those streams began with the same 4 songs in the same order. (4 marks)

Solution
Number of different arrangements for first 4 songs: $${}^{15}P_4 = 32\,760$$ Using the pigeonhole principle, we have $75\,000$ pigeons to place in $32\,760$ pigeonholes. $$\lceil 75000 \div 32760 \rceil = 3$$ Hence at least 3 of the streams must have begun with the same 4 songs in the same order.
Specific behaviours
✓ number of arrangements ✓ identifies pigeons ✓ identifies pigeonholes ✓ uses pigeonhole principle to draw conclusion

Question 15 **(8 marks)**

(a) State whether each of the following statements are true or false, supporting each answer with an example or counterexample.

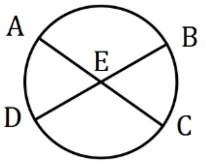
(i) $\forall a, b, c, d \in \mathbb{R}$, if $a < b$ and $c < d$ then $ac < bd$. (2 marks)

Solution
False. Let $a = -2, b = 1$ and $c = -3, d = 0$. Then $a < b$ and $c < d$ but $ac = 6$ and $bd = 0$ and so $ac > bd$.
Specific behaviours
✓ states false ✓ valid counterexample

(ii) $\forall n \in \mathbb{N}$, $n^2 = 2 \times {}^nC_2 + {}^nC_1$. (2 marks)

Solution
True. When $n = 2$, $LHS = 2^2 = 4$ $RHS = 2 \times {}^2C_2 + {}^2C_1 = 2 \times 1 + 2 = 4 = LHS$ as required.
Specific behaviours
✓ states true ✓ valid example using a natural number (must be ≥ 2)

(b) Prove by contradiction that $ABCD$ is not a cyclic quadrilateral if diagonal AC of length 13 cm cuts diagonal BD of length 12 cm at E so that $AE = DE = 4$ cm. (4 marks)

Solution
Assume that $ABCD$ is a cyclic quadrilateral, as shown below:  $CE = 13 - 4 = 9$ cm and $BE = 12 - 4 = 8$ cm. By the intersecting chord theorem, $AE \times CE = BE \times DE$ However, $AE \times CE = 4 \times 9 = 36$ but $BE \times DE = 8 \times 4 = 32$ which contradicts our initial assumption and so $ABCD$ is not a cyclic quadrilateral.
Specific behaviours
✓ states assumption that quadrilateral is cyclic ✓ calculates correct segment lengths ✓ uses intersecting chord theorem ✓ indicates contradiction

Question 16 **(9 marks)**

Office buildings are heated during cold winter days by powerful air-conditioning units. Although the temperature of individual offices rises quickly and stays the same during business hours, the actual concrete and metal structure of the building takes time to completely heat and cool again.

During a cold winter day, the air-conditioning system is powered up at $7am$ before all office staff arrive, and then it is turned off later in the afternoon. The temperature $T(t)$ of the office building, in degrees Celsius, for any time t in hours from midnight, can be modelled by a sinusoid according to the rule:

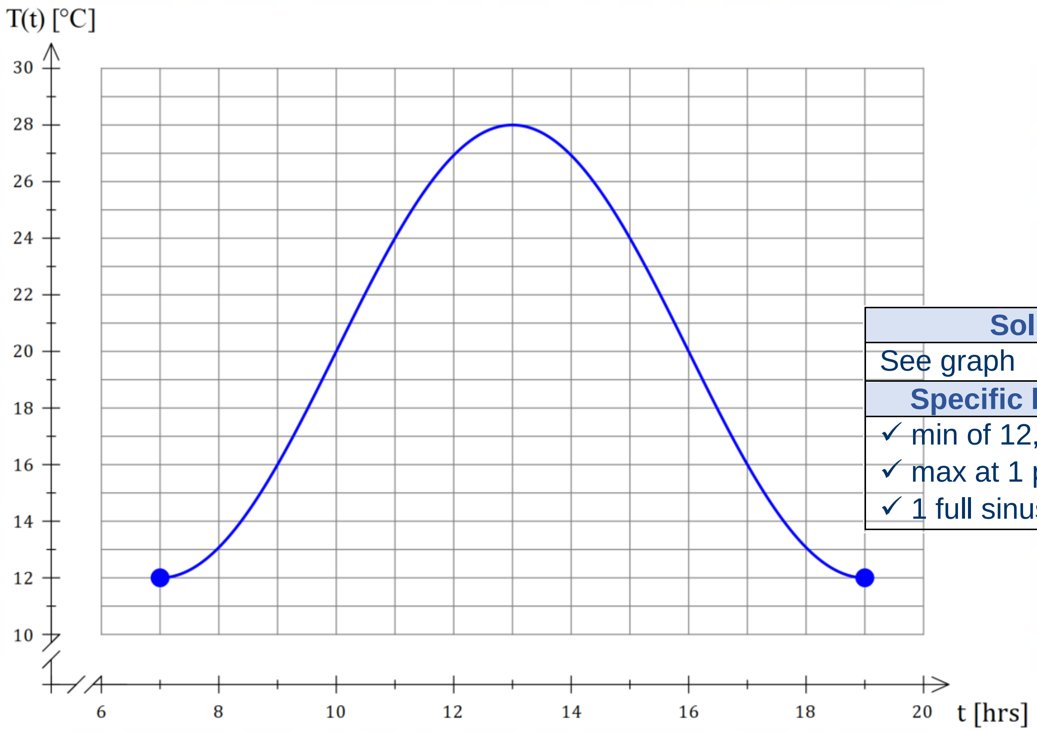
$$T(t) = v - 8\cos(\omega t + \theta)$$

The coldest temperature that the building reaches is $12°C$ and it occurs at $7am$ and $7pm$.

(a) Determine the value of v, ω and θ, and hence state the function for $T(t)$. (4 marks)

Solution	Specific behaviours
$T_{min} = v - 8 = 12 \Rightarrow v = 20$ $Period = 12 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{6}$ $phase\ shift \Rightarrow T(t) = 20 - 8\cos\left[\frac{\pi}{6}(t-7)\right]$ $\Rightarrow \theta = -\frac{7\pi}{6}$ i.e. $T(t) = 20 - 8\cos\left[\frac{\pi}{6}t - \frac{7\pi}{6}\right]$	✓ states value of v ✓ states value of ω ✓ states value of θ ✓ expresses function $T(t)$ (allow either form of function)

(b) Sketch the graph of $T(t)$ on the grid below for the domain $7 \le t \le 19\ hrs$. (3 marks)



Solution
See graph
Specific behaviours
✓ min of 12, max of 28
✓ max at 1 pm (i.e. t=13)
✓ 1 full sinusoidal cycle

(c) Determine the proportion of time over the given domain when the temperature of the building is above $24°C$. (2 marks)

Solution
$20 - 8\cos\left[\frac{\pi}{6}(t-7)\right] = 24 \quad \therefore t = 11\ or\ 15 \quad \Rightarrow$ 4 hours   Proportion of time: $\frac{4}{19-7} = \frac{1}{3} = 0.\overline{3}$
Specific behaviours
✓ determines times when temperature is 24 degrees   ✓ calculates proportion out of the 12 hour domain

Question 17 **(8 marks)**

(a) 5 students from Class A, 8 from Class B and 10 from Class C have nominated for the 3 places available in the team for a mathematics competition. Determine the number of different teams that can be formed if

(i) the students are chosen from the same class. (2 marks)

Solution
$\binom{5}{3}+\binom{8}{3}+\binom{10}{3}=10+56+120=186\text{ teams}$
Specific behaviours
✓ uses combinations ✓ correct number

(ii) at least 2 students in the team are chosen from Class A. (2 marks)

Solution
$\binom{5}{2}\binom{18}{1}+\binom{5}{3}\binom{18}{0}=180+10=190\text{ teams}$
Specific behaviours
✓ identifies both cases ✓ correct number

(b) Prove that for $n \geq 5$, ${}^{n}C_4 + {}^{n}C_5 = {}^{n+1}C_5$. (4 marks)

Solution

$$\begin{aligned}
LHS &= {}^{n}C_4 + {}^{n}C_5 \\
&= \frac{n!}{4!\,(n-4)!} + \frac{n!}{5!\,(n-5)!} \\
&= \frac{5\times n!}{5\times 4!\,(n-4)!} + \frac{(n-4)n!}{5!\,(n-4)(n-5)!} \\
&= \frac{5n!}{5!\,(n-4)!} + \frac{n\cdot n! - 4n!}{5!\,(n-4)!} \\
&= \frac{n! + n\cdot n!}{5!\,(n-4)!} \qquad (**) \\
&= \frac{(n+1)!}{5!\,(n+1-5)!} \\
&= {}^{n+1}C_5 \\
&= RHS
\end{aligned}$$

Specific behaviours

✓ expresses LHS using factorials
✓ obtains common denominator
✓ simplifies to single fraction - $(**)$ line above
✓ completes proof

Question 18 **(3 marks)**

Let $B = \begin{bmatrix} -1 & 5 \\ 2 & -8 \end{bmatrix}$ and $C = \begin{bmatrix} 7 \\ -11 \end{bmatrix}$. Determine X when $X - 5BC = B^2X$.

Solution
$X - B^2X = 5BC$ $(I - B^2)X = 5BC$ $X = (I - B^2)^{-1} \times 5BC$ $I - B^2 = \begin{bmatrix} -10 & 45 \\ 18 & -73 \end{bmatrix}$, $(I - B^2)^{-1} = \begin{bmatrix} 73/80 & 9/16 \\ 9/40 & 1/8 \end{bmatrix}$, $5BC = \begin{bmatrix} -310 \\ 510 \end{bmatrix}$ $X = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$
Specific behaviours
✓ indicates (post) factoring of X ✓ indicates correct equation for X ✓ correct matrix X

Alternate Solution
$X - 5 \times \begin{bmatrix} -1 & 5 \\ 2 & -8 \end{bmatrix} \times \begin{bmatrix} 7 \\ -11 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 2 & -8 \end{bmatrix}^2 \times X$ $\Rightarrow X - \begin{bmatrix} -310 \\ 510 \end{bmatrix} = \begin{bmatrix} 11 & -45 \\ -18 & 74 \end{bmatrix} \times X$ $\left(I - \begin{bmatrix} 11 & -45 \\ -18 & 74 \end{bmatrix}\right)X = \begin{bmatrix} -310 \\ 510 \end{bmatrix}$ $\Rightarrow X = \left(I - \begin{bmatrix} 11 & -45 \\ -18 & 74 \end{bmatrix}\right)^{-1} \times \begin{bmatrix} -310 \\ 510 \end{bmatrix}$ $\therefore X = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$
Specific behaviours
✓ evaluates BC and B^2 ✓ factors out X and uses inverse ✓ solves correct matrix X

Question 19 **(8 marks)**

(a) Determine each of the following in terms of n.

(i) $^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n \quad =$ (1 mark)

Solution
$2^n - 1$
Specific behaviours
✓ correct expression

(ii) $^nC_{n-1} \times {}^{n+1}C_0 \quad =$ (2 marks)

Solution
$^nC_{n-1} = n$ is the penultimate element in row n $^{n+1}C_0 = 1$ is the first element in row $n+1$ $\therefore {}^nC_{n-1} \times {}^{n+1}C_0 = n \times 1 = n$ (or it can be done algebraically using factorials)
Specific behaviours
✓ identifies value of each term ✓ correct product

(b) For each of the following, write expressions using the notation ^nC_r and/or ^nP_r as needed. Do **not** evaluate your answer.

(i) How many 5-character passwords can be created from the lower-case letters of the alphabet, without repetition, that contain exactly three vowels? (2 mark)

Solution
$\binom{5}{3} \times \binom{21}{2} \times 5! \qquad (= 252\,000)$
Specific behaviours
✓ uses combinations to select vowels and consonants ✓ multiplies factorial term (allow 1 mark if *evaluated answer only*)

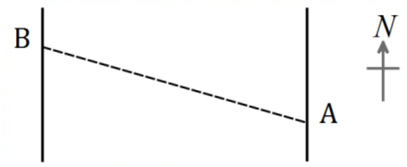
(iii) How many 8-character passwords can be created from the lower-case letters of the alphabet and the digits 0 to 9, without repetition, that contain exactly three vowels OR exactly three consonants?

(4 marks)

Solution
$n(3\ vowels) = {}^5C_3 \times {}^{31}C_5 \times 8!$ $n(3\ consonants) = {}^{21}C_3 \times {}^{15}C_5 \times 8!$ $n(3\ vowels\ AND\ 3\ consonants) = {}^5C_3 \times {}^{21}C_3 \times {}^{10}C_2 \times 8!$ $n(3\ vowels\ OR\ 3\ consonants) = n(3v) + n(3c) - n(3v \cap 3c)$ $= ({}^5C_3 \times {}^{31}C_5 + {}^{21}C_3 \times {}^{15}C_5 - {}^5C_3 \times {}^{21}C_3 \times {}^{10}C_2) \times 8!$
Specific behaviours
✓ expression for exactly 3 vowels ✓ expression for exactly 3 consonants ✓ expression for exactly 3 vowels AND 3 consonants ✓ applies inclusion-exclusion principle for the ‘or’ case

Question 20 **(9 marks)**

Points A and B lie on opposite sides of a river so that B is 320 m away from A on a bearing of $290°$.

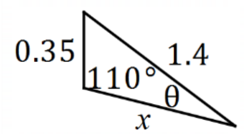


A uniform current flows due south in the river between A and B at 0.35 m/s.

Riley can swim at a steady speed of 1.4 m/s and plans to swim from A to B and then back to A.

(a) Determine the bearing Riley should swim to move directly towards B from A. (3 marks)

Solution



$$\frac{\sin\theta}{0.35} = \frac{\sin 110°}{1.4} \Rightarrow \theta = 13.6°$$

Bearing: $290° + 14° = 304°$

Specific behaviours

✓ diagram with angle
✓ equation using sine rule
✓ correct bearing

(b) Show that Riley takes 42 seconds less to swim the return leg than the first leg. (6 marks)

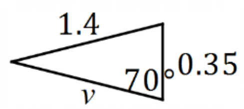
Solution

Speed across ground from A to B:

$$\frac{\sin(180° - 110° - 13.6°)}{x} = \frac{\sin 110°}{1.4} \Rightarrow x = 1.241 \text{ m/s}$$

Time $AB = 320 \div 1.241 = 258$ s

Return leg from B to A:



$$1.4^2 = v^2 + 0.35^2 - 2(0.35)v\cos 70° \Rightarrow v = 1.481 \text{ m/s}$$

Time $BA = 320 \div 1.481 = 216$ s

Hence $258 - 216 = 42$ second less.

Specific behaviours

✓ speed from A to B
✓ time from A to B
✓ diagram for B to A
✓ speed from B to A
✓ time from B to A
✓ confirms difference

Question 21 **(8 marks)**

A common proof that $\sqrt[4]{3}$ is irrational begins by assuming that $\sqrt[4]{3}$ is rational, so that $\sqrt[4]{3} = \dfrac{a}{b}$.

(a) Describe two properties of variables a and b that the proof requires, other than $b \neq 0$. (2 marks)

Solution
a and b are integers and have no common factor.
Specific behaviours
✓ states both are integers ✓ states no common factor, divisor, co-prime etc.

The next step obtains the relationship $a^4 = 3b^4$, from which it is deduced that $a = 3A, A \in \mathbb{Z}$.

(b) Prove, using the contrapositive, that if a^4 is a multiple of 3 then so is a. (4 marks)

Solution
Contrapositive: If a is not a multiple of 3 then neither is a^4. If a is not a multiple of 3, then it must be of the form $3k+1$ or $3k+2$ $(k \in \mathbb{Z})$. Case 1: $a = 3k+1$, $a^4 = 81k^4 + 108k^3 + 54k^2 + 12k + 1 = 3(27k^4 + 36k^3 + 18k^2 + 4k) + 1$ i.e. not a multiple of 3 Case 2: $a = 3k+2$, $a^4 = 81k^4 + 216k^3 + 216k^2 + 96k + 16 = 3(27k^4 + 72k^3 + 72k^2 + 32k + 5) + 1$ i.e. not a multiple of 3 It can be seen in each case that a^4 is not an integer multiple of 3. As the contrapositive is true then the original statement must be true.
Specific behaviours
✓ writes contrapositive ✓ identifies cases for a in terms of some constant integer (deduct mark if no "$k \in \mathbb{Z}$") ✓ shows a^4 is not multiple of 3 for one case ✓ shows a^4 is not multiple of 3 for other case and concludes

(c) Complete the proof that $\sqrt[4]{3}$ is irrational. (2 marks)

Solution
Since $a = 3A$ then $a^4 = 3b^4 \Rightarrow (3A)^4 = 3b^4 \Rightarrow b^4 = 3(9A^4)$. Thus b^4 and b are also multiples of 3. Hence a and b are both multiples of 3 - a contradiction of the initial assumption and so $\sqrt[4]{3}$ is irrational.
Specific behaviours
✓ deduces that b is multiple of 3 ✓ indicates contradiction

End of questions

Supplementary page

Question number: _________

Supplementary page

Question number: _________

Supplementary page

Question number: _________