Rossmoyne Senior High School

Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	94	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

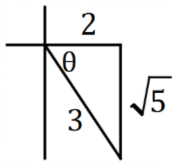
Question 1 **(6 marks)**

(a) State the exact value of $\cot 60°$. (1 mark)

Solution
$\dfrac{1}{\tan 60°}=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$
Specific behaviours
✓ correct value (must rationalise denominator)

(b) Given that $\sec\theta=\dfrac{3}{2}$ and $-90°\leq\theta\leq 0°$, state the exact value of $\operatorname{cosec}\theta$. (2 marks)

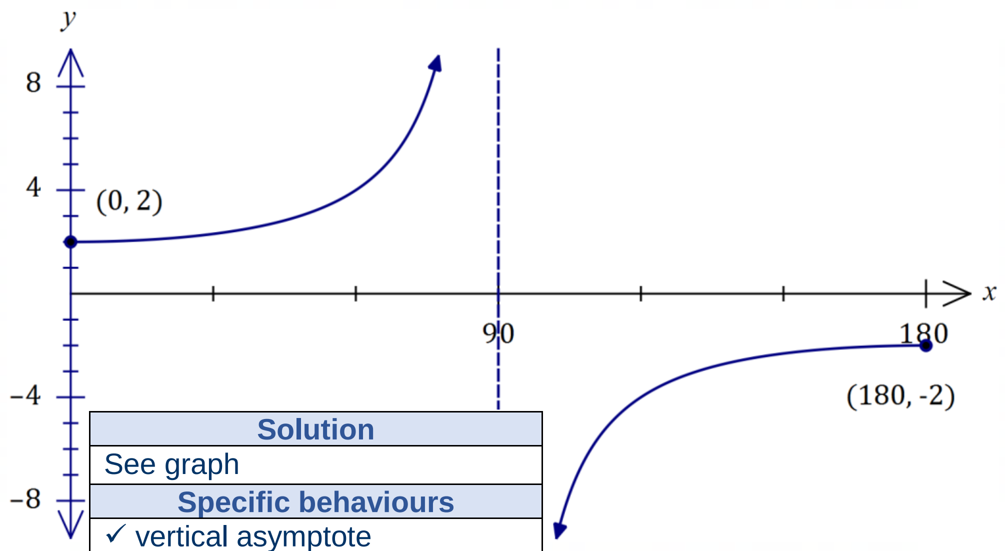
Solution



$$\operatorname{cosec}\theta=\frac{3}{-\sqrt{5}}=-\frac{3\sqrt{5}}{5}$$

Specific behaviours
✓ value is negative
✓ correct value (allow if not rationalised denominator)

(c) Sketch the graph of $y=2\sec x$ for $0°\leq x\leq 180°$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ vertical asymptote
✓ correct endpoints
✓ smooth curves (not too steep)

Question 2 **(7 marks)**

Two matrices are $A = \begin{bmatrix} 5 & -5 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -4 \\ -2 & 2 \end{bmatrix}$. Determine

(a) $4A - 5B$. (2 marks)

Solution
$4A - 5B = \begin{bmatrix} 20 & -20 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 25 & -20 \\ -10 & 10 \end{bmatrix}$ $= \begin{bmatrix} -5 & 0 \\ 10 & -2 \end{bmatrix}$
Specific behaviours
✓ one correct multiple   ✓ correct matrix

(b) B^{-1}. (2 marks)

Solution
$\lvert B \rvert = (5)(2) - (-2)(-4) = 2$ $B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2.5 \end{bmatrix}$
Specific behaviours
✓ determinant   ✓ correct matrix

(c) $AB + A^2$. (3 marks)

Solution
$A(B + A) = \begin{bmatrix} 5 & -5 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 10 & -9 \\ -2 & 4 \end{bmatrix}$ $= \begin{bmatrix} 60 & -65 \\ -4 & 8 \end{bmatrix}$
Specific behaviours
✓ factors   ✓ $B + A$ ✓ correct matrix

Alternative Solution
$AB = \begin{bmatrix} 35 & -30 \\ -4 & 4 \end{bmatrix}$ $A^2 = \begin{bmatrix} 25 & -35 \\ 0 & 4 \end{bmatrix}$ $AB + A^2 = \begin{bmatrix} 60 & -65 \\ -4 & 8 \end{bmatrix}$
Specific behaviours
✓ AB ✓ B^2 ✓ correct matrix

Question 3 **(8 marks)**

(a) Express $\left(\sqrt{7}+\sqrt{-7}\right)^2$ in the form $a+bi$ where $a,b \in \mathbb{R}$. (2 marks)

Solution
$(\sqrt{7}+\sqrt{-7})(\sqrt{7}+\sqrt{-7}) = 7+2\sqrt{7}\sqrt{7}i-7$ $= 14i$ OR $\left[\sqrt{7}(1+i)\right]^2 = 7(1+2i-1) = 14i$
Specific behaviours
✓ uses $i^2=-1$ ✓ correctly simplifies

(b) Two complex numbers are $u = 6-2i$ and $v = 2+2i$. Calculate

(i) $u \times v$. (1 mark)

Solution
$uv = (6-2i)(2+2i)$ $= 16+8i$
Specific behaviours
✓ correct product

(ii) $u \div v$. (2 marks)

Solution
$\frac{u}{v} = \frac{6-2i}{2+2i} \times \frac{2-2i}{2-2i}$ $= \frac{8-16i}{8}$ $= 1-2i$
Specific behaviours
✓ correctly uses conjugate of v (or uses $\frac{1-i}{1-i}$)   ✓ correct quotient, simplified

(iii) $\mathrm{Im}(4\bar{v} - iu)$. (3 marks)

Solution
$4\bar{v} = 4(2-2i) = 8-8i$ $iu = i(6-2i) = 2+6i$ $4\bar{v} - iu = 6-14i \Rightarrow \mathrm{Im}(4\bar{v}-iu) = -14$
Specific behaviours
✓ $4\bar{v}$ ✓ product iu ✓ imaginary part of difference

Question 4 **(8 marks)**

(a) Use an angle sum identity to prove that $\sin 2A = 2\sin A\cos A$. (2 marks)

Solution
$LHS = \sin(2A) = \sin(A + A)$ $= \sin A\cos A + \cos A\sin A$ $= 2\sin A\cos A$ $= RHS$
Specific behaviours
✓ uses angle sum identity ✓ simplifies

(b) Hence, or otherwise, prove that $\sin 3A = 3\sin A - 4\sin^3 A$. (3 marks)

Solution
$LHS = \sin 3A = \sin(2A + A)$ $= \sin 2A\cos A + \cos 2A\sin A$ $= (2\sin A\cos A)\cos A + (1 - 2\sin^2 A)\sin A$ $= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$ $= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$ $= 3\sin A - 4\sin^3 A$ $= RHS$
Specific behaviours
✓ uses angle sum identity ✓ uses double angle identities ✓ simplifies

(c) Find all solutions to $3\sin A = 1 + 4\sin^3 A$ giving your answer in degrees. (3 marks)

Solution
$3\sin A - 4\sin^3 A = 1$ $\sin 3A = 1$ $3A = 90 + 360n \quad n \in \mathbb{Z}$ $\Rightarrow A = 30 + 120n \quad n \in \mathbb{Z}$
Specific behaviours
✓ writes equation using triple angle ✓ correct solution for $3A$ (with the $360n$) ✓ correct solution for A *deduct 1 mark if n is not defined

Question 5 **(6 marks)**

(a) Determine the equation of the real quadratic $f(z)$ in the form $z^2 + az + b$ given that $f(4 + 6i) = 0$. (2 marks)

Solution
$f(z) = [z - (4+6i)][z - (4-6i)]$ $= z^2 - 8z + 52$
Specific behaviours
✓ shows product of linear factors   ✓ correct equation

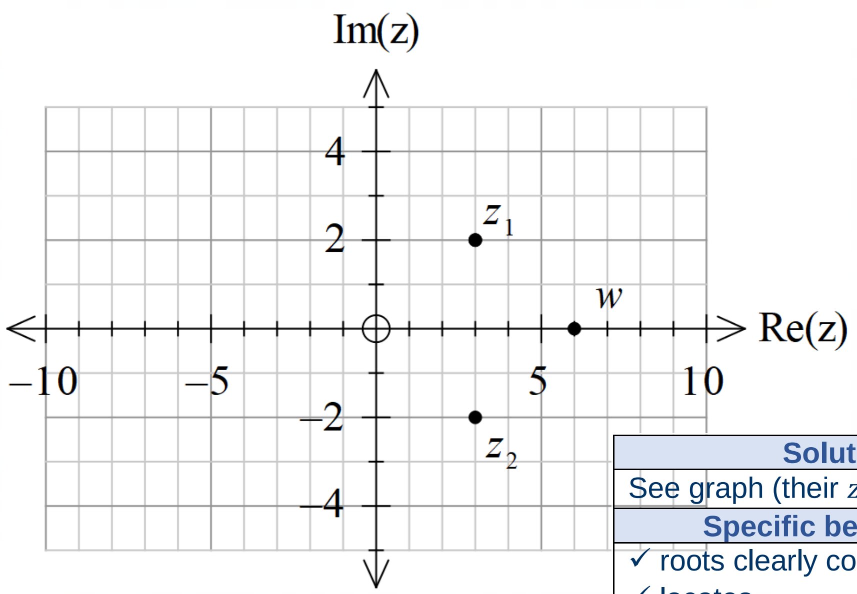
Alternative Solution
$a = -2(4) = -8$ $b = 4^2 + 6^2 = 52$ $f(z) = z^2 - 8z + 52$
Specific behaviours
✓ shows sum and product of roots   ✓ correct equation

(b) Let $g(z) = z^2 - 6z + 13$.

(i) Determine z_1 and z_2, the complex roots of g. (2 marks)

Solution
$(z-3)^2 = -13 + 3^2$ $= -4 = (2i)^2$ $z = 3 \pm 2i$ $z_1 = 3 + 2i, \quad z_2 = 3 - 2i$
Specific behaviours
✓ indicates suitable method, e.g. quadratic formula   ✓ correct conjugate roots

(ii) Sketch and label z_1, z_2 and $w = z_1 + z_2$ in the complex plane below. (2 marks)



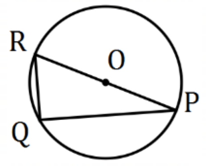
Solution
See graph (their z_1 and z_2)
Specific behaviours
✓ roots clearly conjugate pair   ✓ locates w

Question 6 **(5 marks)**

PR is a diameter of a circle centre O and point Q lies on the circumference of the circle.

Let $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$.

Use a vector method to prove that $\angle PQR = 90°$.

Solution
 $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}, \qquad \overrightarrow{RQ} = \mathbf{p} + \mathbf{q}$ $\overrightarrow{RQ} \cdot \overrightarrow{PQ} = (\mathbf{p} + \mathbf{q}) \cdot (\mathbf{q} - \mathbf{p})$ $= \mathbf{p} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - \mathbf{q} \cdot \mathbf{p}$ $= \lvert\mathbf{q}\rvert^2 - \lvert\mathbf{p}\rvert^2$ $= 0 \quad (\lvert\mathbf{p}\rvert = \lvert\mathbf{q}\rvert = \text{radius})$ Hence $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$ are perpendicular ($\angle PQR = 90°$) since their magnitudes are not zero yet the scalar product is zero.
Specific behaviours
✓ labelled sketch ✓ vectors $\overrightarrow{PQ}$ and $\overrightarrow{RQ}$ ✓ forms scalar product ✓ simplifies to 0 with explanation $\lvert\mathbf{p}\rvert = \lvert\mathbf{q}\rvert = \text{radius}$ ✓ explains result, including that magnitudes are non-zero *deduct 1 mark if missing/incorrect vector or dot product notation

Question 7 **(6 marks)**

(a) Determine the vector projection of $2\mathbf{i} - \mathbf{j}$ on $-3\mathbf{i} + 4\mathbf{j}$. (2 marks)

Solution
$\left(\frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\end{pmatrix}\right)\frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix} = \frac{-10}{25}\begin{pmatrix}-3\\4\end{pmatrix} = \begin{pmatrix}6/5\\-8/5\end{pmatrix}$
Specific behaviours
✓ indicates method
✓ correct projection

Alternative Solution
$\dfrac{\begin{pmatrix}2\\-1\end{pmatrix}\cdot\begin{pmatrix}-3\\4\end{pmatrix}}{\begin{pmatrix}-3\\4\end{pmatrix}\cdot\begin{pmatrix}-3\\4\end{pmatrix}}\begin{pmatrix}-3\\4\end{pmatrix} = \frac{-10}{25}\begin{pmatrix}-3\\4\end{pmatrix} = \begin{pmatrix}6/5\\-8/5\end{pmatrix}$
Specific behaviours
✓ indicates method
✓ correct projection

(b) The vectors $a\mathbf{i} + \mathbf{j}$ and $4\mathbf{i} + b\mathbf{j}$ are perpendicular and the sum of their magnitudes is 10. Determine the values of the constants a and b. (4 marks)

Solution
Perpendicular: $4a + b = 0$.
Magnitudes: $\sqrt{a^2+1} + \sqrt{16+b^2} = 10$.
Hence
$b = -4a \Rightarrow b^2 = 16a^2$
$\sqrt{a^2+1} + \sqrt{16+16a^2} = 10$
$\sqrt{a^2+1} + 4\sqrt{a^2+1} = 10$
$\sqrt{a^2+1} = 2$
$a^2 = 3 \Rightarrow a = \pm\sqrt{3}$
$b = \mp 4\sqrt{3}$
Hence $a = \sqrt{3}, b = -4\sqrt{3}$ or $a = -\sqrt{3}, b = 4\sqrt{3}$.
Specific behaviours
✓ two equations
✓ eliminates one variable
✓ solves for one variable
✓ clearly states both solution pairs

Question 8 **(6 marks)**

Use mathematical induction to prove the following proposition $P(n)$ for every integer $n \geq 0$.

$$P(n):\ 1 + 7 + 13 + 19 + \cdots + (6n + 1) = (n + 1)(3n + 1)$$

Solution
1. When $n = 0$: $P(0)$: $LHS = 6(0) + 1 = 1$ $RHS = (0 + 1)(3(0) + 1) = 1 = LHS$ $\therefore$ true for n=0 2. Assume $P(k)$ is true: $\Rightarrow 1 + 7 + \cdots + (6k + 1) = (k + 1)(3k + 1)$ 3. Now required to prove $P(k + 1)$: $RTP: 1 + 7 + \cdots + (6k + 1) + (6(k + 1) + 1) = \big((k + 1) + 1\big)(3(k + 1) + 1)$ $= (k + 2)(3k + 4)$ $LHS = 1 + 7 + \cdots + (6k + 1) + (6(k + 1) + 1)$ $= (k + 1)(3k + 1) + (6k + 7)$ $= 3k^2 + 4k + 1 + 6k + 7$ $= 3k^2 + 10k + 8$ $= (k + 2)(3k + 4)$ $= RHS$ $\therefore$ true for n=k+1 Since $P(0)$ is true, and we have shown that $P(k + 1)$ is true if $P(k)$ is true, then $P(n)$ is true for all integers $n \geq 0$.
Specific behaviours
✓ shows true for $n = 0$ ✓ states assumption that $P(k)$ is true ✓ substitutes the assumption into the $n = k + 1$ case/expression ✓ expands LHS ✓ factors LHS to required form ✓ uses principle of mathematical induction, i.e. concluding statement

End of questions

Supplementary page

Question number: _________