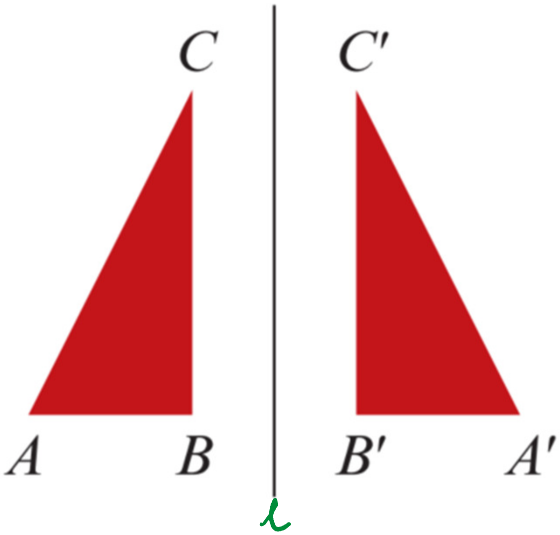
16B Geometric transformations

– transformations that are geometric in nature

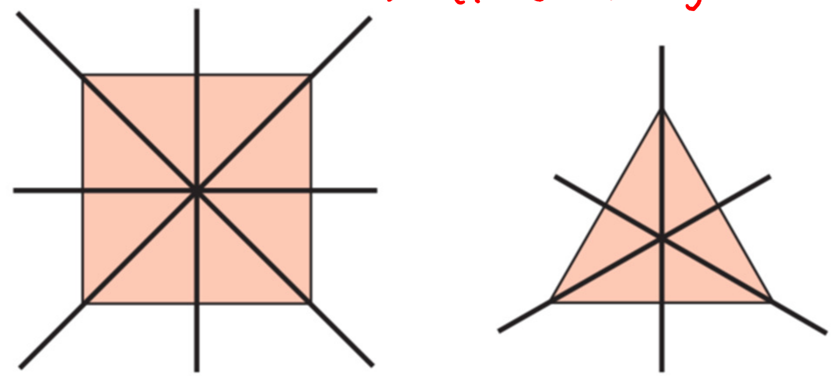
It is space itself that gets transformed.



points and their image are the same distance from line l and

AA' is $\perp$ to L

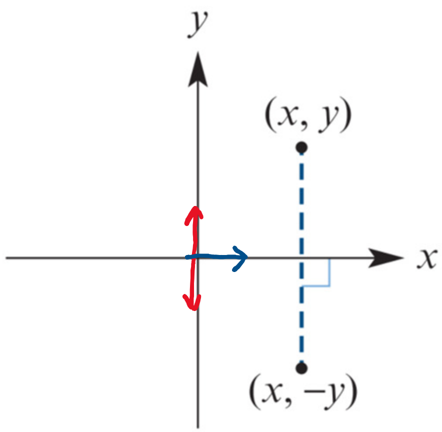
– line of symmetry



Geometric transformations are important in studying figures with reflective symmetry

An isometry is a transformation that does not change lengths.

Reflection in the x-axis



$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ -y \end{bmatrix}$$

remember we investigate transformations by looking at their effect on $\underset{\sim}{i}$ and $\underset{\sim}{j}$

change of basis

$$\underset{\underset{\sim}{i}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \rightarrow \underset{\underset{\sim}{i}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$\underset{\underset{\sim}{j}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \rightarrow \underset{-\underset{\sim}{j}}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

so the transformation we need is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

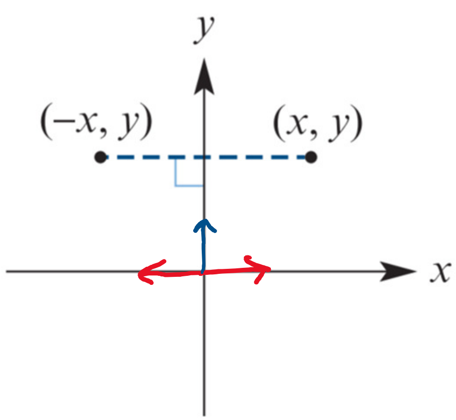
so $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underset{\text{image}}{\begin{bmatrix} x \\ -y \end{bmatrix}}$

to get the image $\begin{bmatrix} x' \\ y' \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = 1x + 0y$$
$$y' = 0x - 1y$$

Reflection in the ~~x~~ y-axis



$$\begin{bmatrix}x\\y\end{bmatrix} \to \begin{bmatrix}-x\\y\end{bmatrix}$$

$$\underset{\underset{\sim}{i}}{\begin{bmatrix}1\\0\end{bmatrix}} \to \underset{-\underset{\sim}{i}}{\begin{bmatrix}-1\\0\end{bmatrix}}$$

$$\underset{\underset{\sim}{j}}{\begin{bmatrix}0\\1\end{bmatrix}} \to \underset{\underset{\sim}{j}}{\begin{bmatrix}0\\1\end{bmatrix}}$$

remember we investigate transformations by looking at their effect on $\underset{\sim}{i}$ and $\underset{\sim}{j}$

so the transformation we need is $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

so $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \underset{\text{image}}{\begin{bmatrix}-x\\y\end{bmatrix}}$

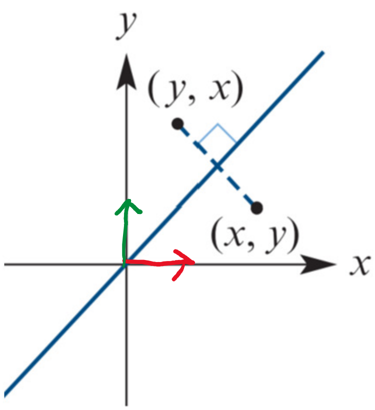
to get the image $\begin{bmatrix}x'\\y'\end{bmatrix}$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$x' = -1x + 0y$$

$$y' = 0x + 1y$$

Reflection in the line $y = x$



$$\begin{bmatrix}1\\0\end{bmatrix} \rightarrow \begin{bmatrix}0\\1\end{bmatrix}$$
$\underset{\sim}{i}$

$$\begin{bmatrix}0\\1\end{bmatrix} \rightarrow \begin{bmatrix}1\\0\end{bmatrix}$$
$\underset{\sim}{j}$

so the transformation we need is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

so $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}$

image

$$\begin{bmatrix}x\\y\end{bmatrix} \rightarrow \begin{bmatrix}y\\x\end{bmatrix}$$

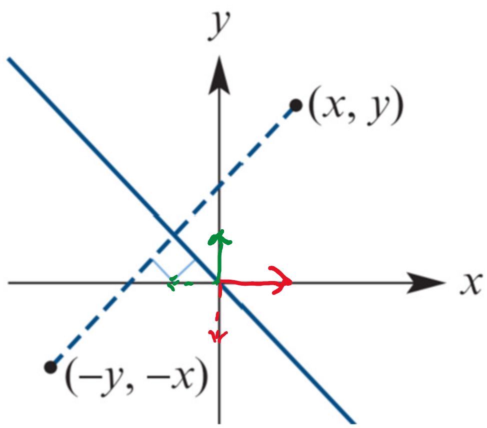
remember we investigate transformations by looking at their effect on $\underset{\sim}{i}$ and $\underset{\sim}{j}$

to get the image $\begin{bmatrix}x'\\y'\end{bmatrix}$

$$\begin{array}{l}y \rightarrow \\ x \rightarrow\end{array}\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$x' = 0x + 1y$$
$$y' = 1x + 0y$$

Reflection in the line $y = \cancel{x}$ $-x$



$$\underset{\underset{\sim}{i}}{\begin{bmatrix}1\\0\end{bmatrix}} \rightarrow \begin{bmatrix}0\\-1\end{bmatrix}$$

$$\underset{\underset{\sim}{j}}{\begin{bmatrix}0\\1\end{bmatrix}} \rightarrow \begin{bmatrix}-1\\0\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix} \rightarrow \begin{bmatrix}-y\\-x\end{bmatrix}$$

remember we investigate transformations by looking at their effect on $\underset{\sim}{i}$ and $\underset{\sim}{j}$

so the transformation we need is $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$

so $\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \underset{\text{image}}{\begin{bmatrix}-y\\-x\end{bmatrix}}$

to get the image $\begin{bmatrix}x'\\y'\end{bmatrix}$

$$\begin{matrix}-y \rightarrow\\-x \rightarrow\end{matrix}\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$x' = 0x - 1y$

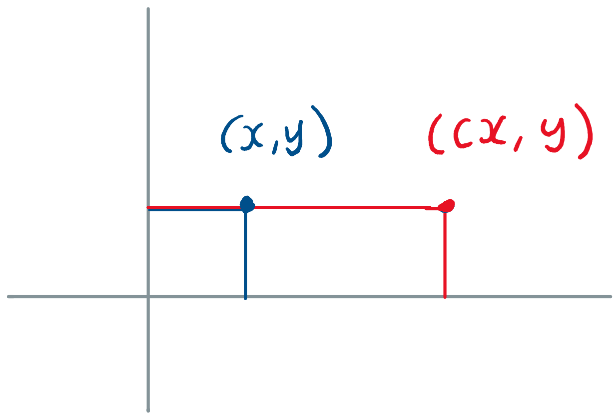
$y' = -1x + 0y$

Transformation	Rule	Matrix
Reflection in the x-axis	$x' = 1x + 0y$ $y' = 0x - 1y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -1x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$x' = 0x + 1y$ $y' = 1x + 0y$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$x' = 0x - 1y$ $y' = -1x + 0y$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

since all vectors in a 2D vector space are defined in terms of $\underline{i}$ and $\underline{j}$, the above matrices can transform any 2D shape by transforming vectors defining its vertices

Dilations

Dilation from the y-axis



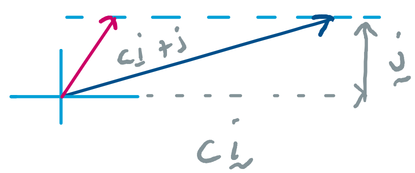
$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} cx \\ y \end{bmatrix}$$

$c > 0$

$$\underset{\sim}{i} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$\underset{\sim}{j} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

y-coordinate (component) unchanged

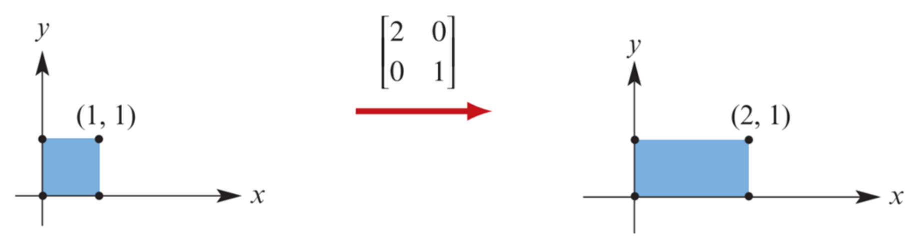


x-coordinate is scaled by a factor of c

The transformation we need is

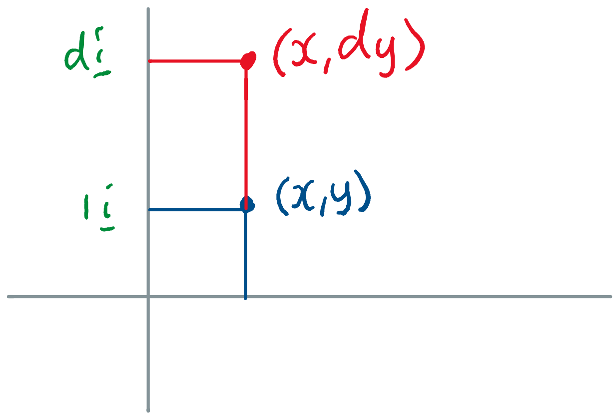
$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$$

so $\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ y \end{bmatrix}$



Dilations

Dilation from the $\overset{x}{\cancel{y}}$-axis



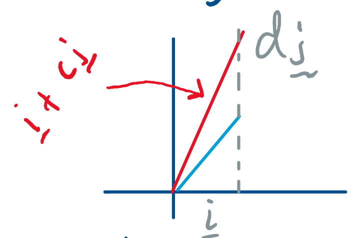
$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ dy \end{bmatrix}$$

$c > 0$

$$\begin{array}{c} \underset{\sim}{i} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \underset{\sim}{j} \end{array} \rightarrow \begin{bmatrix} 0 \\ d \end{bmatrix}$$

x-coordinate (component) unchanged

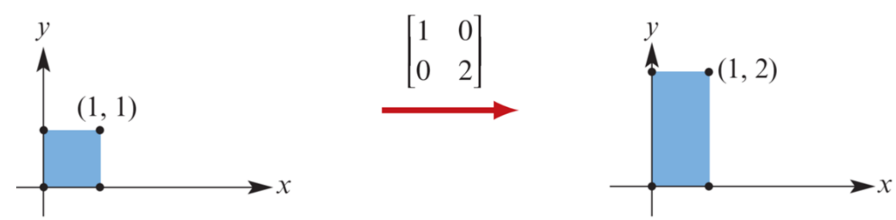


y-coordinate is scaled by a factor of c

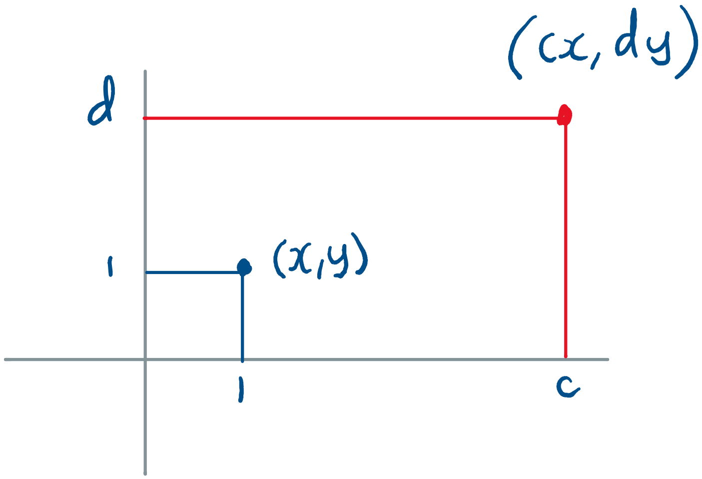
The transformation we need is

$$\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$$

so $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ dy \end{bmatrix}$



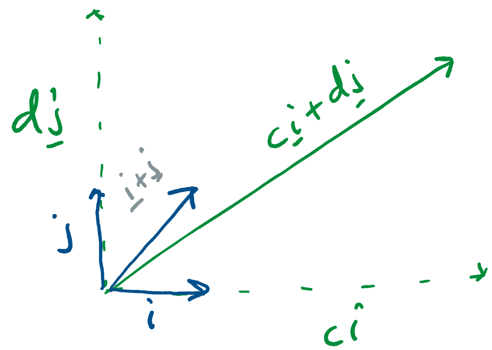
Dilation from the *x*- and *y*-axes



$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} cx \\ dy \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ d \end{bmatrix}$$



The transformation:

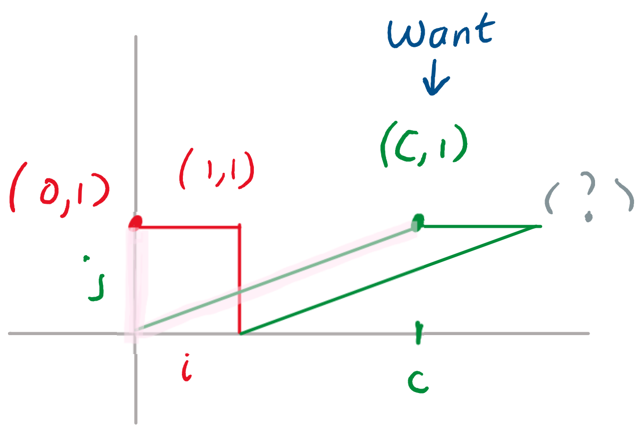
$$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

so $\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ dy \end{bmatrix}$

Transformation	Rule	Matrix
Dilation from the y-axis	$x' = cx + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$
Dilation from the x-axis	$x' = 1x + 0y$ $y' = 0x + cy$	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Dilation from the x- and y-axes	$x' = cx + 0y$ $y' = 0x + dy$	$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

Shears

Shear parallel to the x-axis



$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x + cy \\ y \end{bmatrix}$$

$$\overset{i}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underset{j}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \to \begin{bmatrix} c \\ 1 \end{bmatrix}$$

change of basis

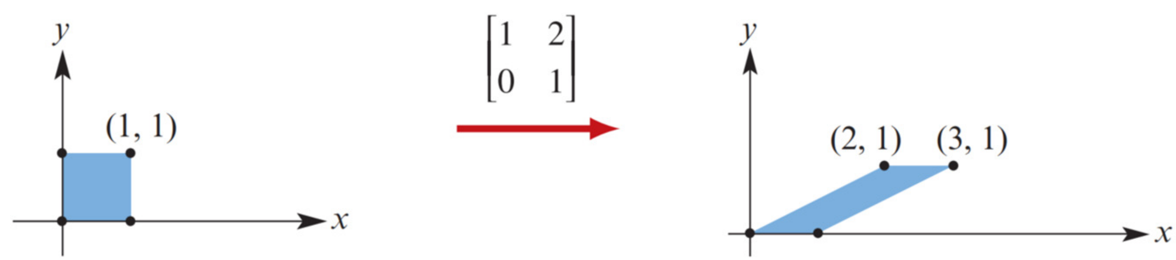
The transformation is

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$$

check point $(1, 1)$

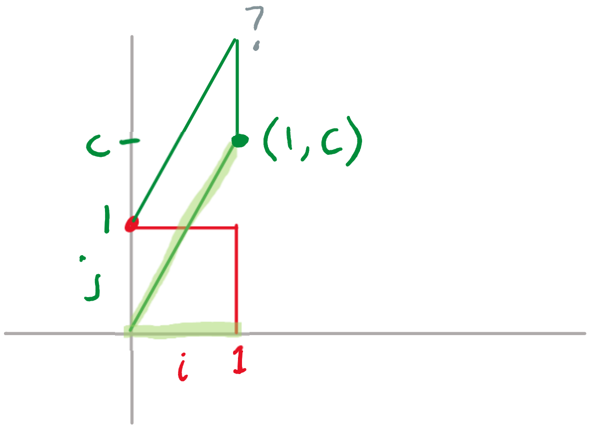
$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+c \\ 1 \end{bmatrix}$$

moves to $(c+1, 1)$
as expected



Shears

Shear parallel to the ~~x~~ y-axis



$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x + cy \\ y \end{bmatrix}$$

$$\overset{i}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \to \begin{bmatrix} 1 \\ c \end{bmatrix}$$

$$\underset{j}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

change of basis

The transformation is

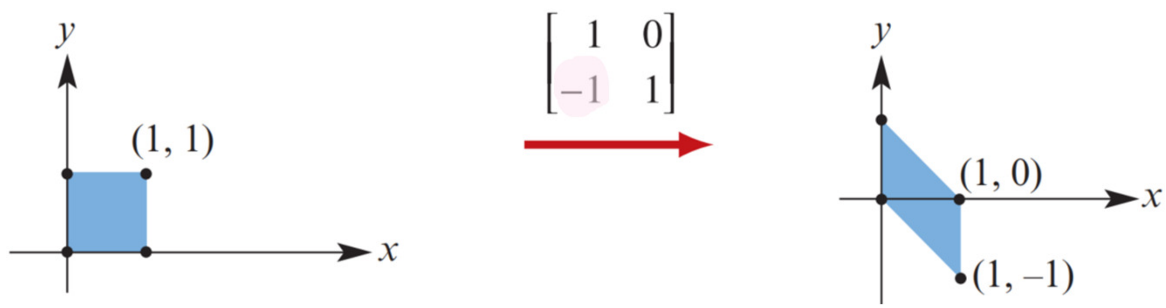
$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

check point (1, 1)

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ c+1 \end{bmatrix}$$

moves to (1, c+1)

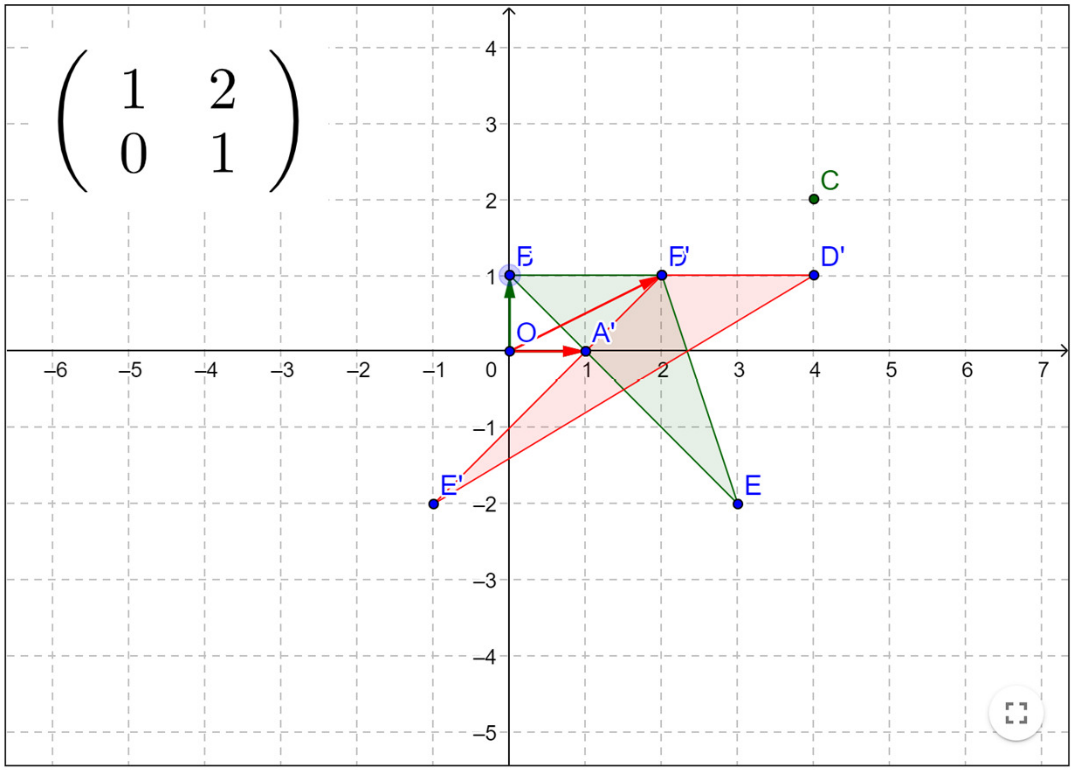
as expected



EXAMPLE 1

The points A(2, 1), B(3, –2) and C(0, 1) are transformed to A′, B′ and C′ by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

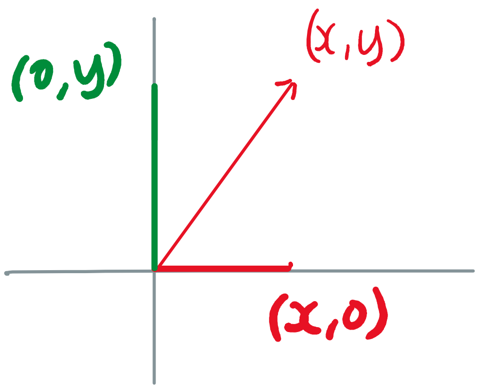
Find the coordinates of A′, B′ and C′.



https://www.geogebra.org/m/qP8UtMVM

Transformation	Rule	Matrix
Shear parallel to the x-axis	$x' = 1x + cy$ $y' = 0x + 1y$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
Shear parallel to the y-axis	$x' = 1x + 0y$ $y' = cx + 1y$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

Projections



$$\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}x\\0\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}0\\0\end{bmatrix}$$

Transformation

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

Projection onto the x-axis

so $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$

$$\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}0\\1\end{bmatrix}$$

Transformation

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

Projection onto the y-axis

so $\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\y\end{bmatrix}$

Transformation	Rule	Matrix
Projection onto the x-axis	$x' = 1x + 0y$ $y' = 0x + 0y$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the y-axis	$x' = 0x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Example 5

Find the image of the point $(3, 4)$ under each of the following transformations:

a reflection in the y-axis

b dilation of factor 2 from the y-axis

c shear of factor 4 parallel to the x-axis

d projection onto the y-axis

Solution

a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$(3, 4) \to (-3, 4)$

b $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$(3, 4) \to (6, 4)$

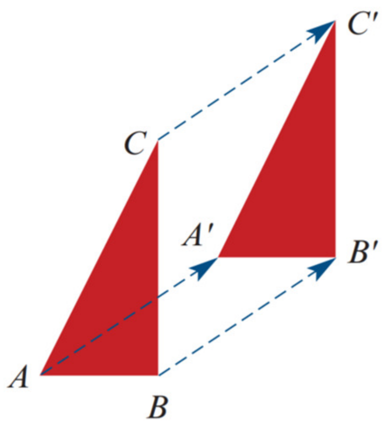
c $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$

$(3, 4) \to (19, 4)$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$(3, 4) \to (0, 4)$

Translations



Every point moves in the same direction and over the same distance

Translate by a units in the x-direction and b units in the y-direction

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+a \\ y+b \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

can not use multiplication.

(0,0) will map to (0,0)

Example 6

Find the rule for a translation of 2 units in the x-direction and -1 units in the y-direction, and sketch the image of the unit square under this translation.

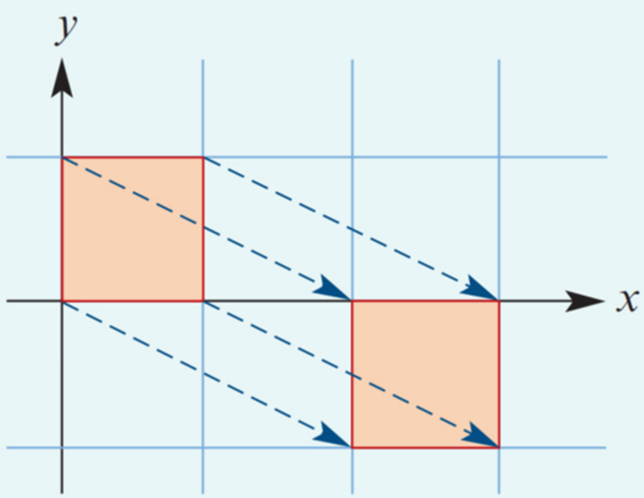
Solution

Using vector addition, this translation can be defined by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x+2 \\ y-1 \end{bmatrix}$$

or equivalently

$$x' = x + 2 \quad \text{and} \quad y' = y - 1$$



Section summary

- Important geometric transformation matrices are summarised in the table below.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- A translation of a units in the x-direction and b units in the y-direction is defined by the rule $(x, y) \to (x + a, y + b)$. This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$