YEAR 11 SPECIALIST: NUMBER AND PROOF

Term 1 Week 2

2B Sets of Numbers

$\sqrt{2}$

Learning Objectives/Syllabus Covered:

2.3.1 prove simple results involving numbers

2.3.2 express rational numbers as terminating or eventually recurring decimals and vice versa

2B Sets of numbers

(Cambridge p43 – 44 included)

Natural numbers $\{1, 2, 3, 4 \ldots\}$ $\mathbb{N}$

Integers $\{\ldots -2, -1, 0, 1, 2 \ldots\}$ $\mathbb{Z}$

Rational numbers → In the form $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$

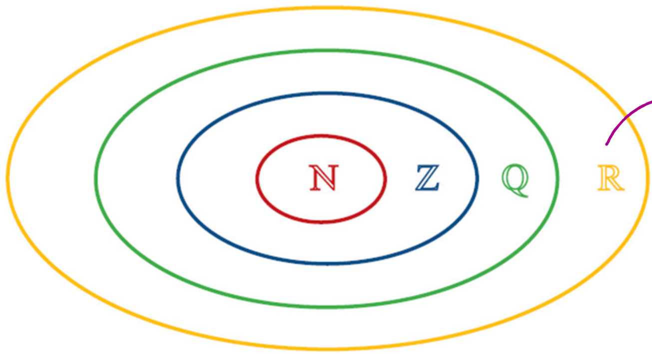
→ Symbol $\mathbb{Q}$

Irrational numbers e.g. $\sqrt{2}, \sqrt{3}, \pi, \pi + 2, \sqrt{6}, \sqrt{7}$

roots of non P.S.

→ can not be written as $\frac{p}{q}$ for $p, q \in \mathbb{Z}$

→ as decimals, do not terminate or repeat



Real numbers
(any positive or negative number)

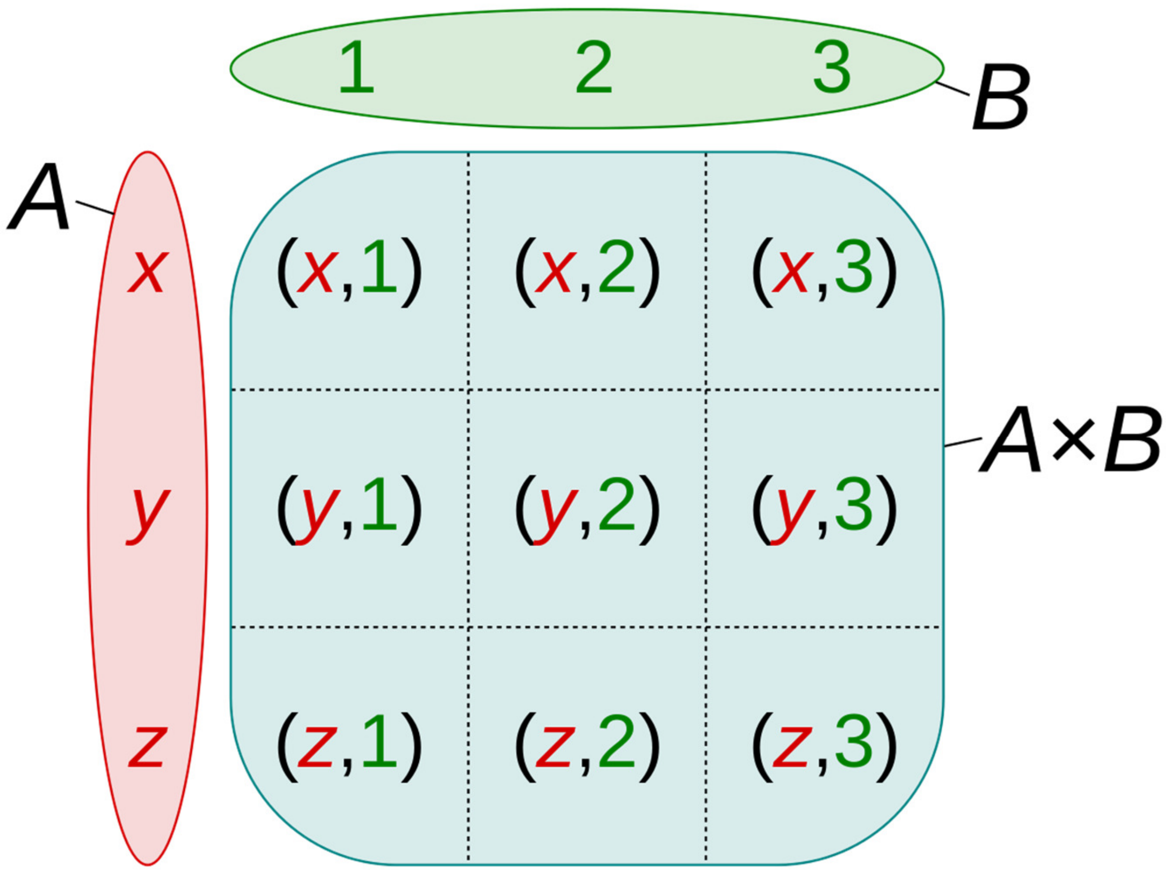
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$\{x : 0 < x < 1\}$ all real numbers strictly btn 0 & 1

$\{x : x > 0, x \in \mathbb{Q}\}$ all positive rational numbers

$\{2n : n = 0, 1, 2 \ldots\}$ ($n \in \mathbb{N} \cup \{0\}$) all non-negative real numbers

$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ all ordered pairs of real numbers
"the Cartesian product of $\mathbb{R}$ with itself



Cartesian product - forms ordered pairs

not in the scope of the course

$A \times B = \{(x,1),(x,2),\ldots (z,2),(z,3)\}$

Rational numbers – can be expressed as terminating or recurring decimals

do $\frac{3}{7}$ as $3.0000\ldots \div 7$

$$\begin{array}{r} 0.\;4\;\;2\;\;8\;\;5\;\;7\;\;1\;\;4\;\;2 \\ 7\overline{)\,3.\,{}^{3}0\,{}^{2}0\,{}^{6}0\,{}^{4}0\,{}^{5}0\,{}^{1}0\,{}^{3}0\,{}^{2}0} \end{array}$$

$$\therefore \quad \frac{3}{7} = 0.\dot{4}2857\dot{1}$$ ← use this

or

$$0.\overline{428571}$$

Theorem

Every rational number can be written as a terminating or recurring decimal.

Proof Consider any two natural numbers m and n. At each step in the division of m by n, there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \ldots, n - 1$. (In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.) Hence the remainder must repeat after at most $n - 1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

Recurring decimals

you saw this in year 8

If we apply the division algorithm to a fraction whose denominator has a prime factor other than 2 or 5, we see that the process does not terminate.

11 Spec

Theorem

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^{\alpha} \times 5^{\beta}}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

Proof Assume that $x = \dfrac{m}{2^\alpha \times 5^\beta}$ with $\alpha \geq \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

not in scope

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so x can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if x can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \dfrac{m}{10^\alpha} = \dfrac{m}{2^\alpha \times 5^\alpha}$.

The method for finding a rational number $\dfrac{m}{n}$ from its decimal representation is demonstrated in the following example.

Example 4

Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.05 **b** $0.\dot{4}2857\dot{1}$

a) $0.05 = \frac{5}{10^2} = \frac{5}{100} = \frac{1}{20}$

$$20 \overline{)1.0\,0}\quad 0.05$$

b) $0.\dot{4}2857\dot{1} = 0.428571\,428571\ldots$ ①

$\times 10^6$ ↑ length of repeat

$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571\ldots$ ②

②−① $0.\dot{4}2857\dot{1} \times 10^6 - 0.\dot{4}2857\dot{1} = 428571$

$0.\dot{4}2857\dot{1}(10^6 - 1) = 428571$

$\therefore\ 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1} = \frac{3}{7}$

Example 4

Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.05 **b** $0.\dot{4}2857\dot{1}$

let $x = 0.\dot{4}2857\dot{1}$

$1{,}000{,}000x = 428571.\dot{4}2857\dot{1}$

$999999x = 428571$

$x = \frac{428571}{999999}$

$= \frac{3}{7}$

simpler way

Calc simplify

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6C Proof by Contradiction

$\sqrt{2}$

Learning Objectives/Syllabus Covered:

1.3.2 use proof by contradiction

2.3.3 prove irrationality by contradiction for numbers such as $\sqrt{2}$

6C Proof by contradiction

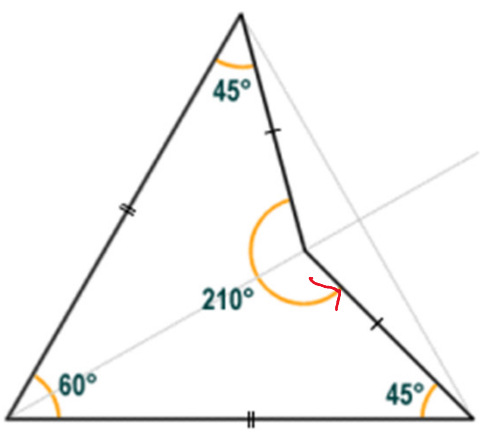
→ Prove something that can not be done

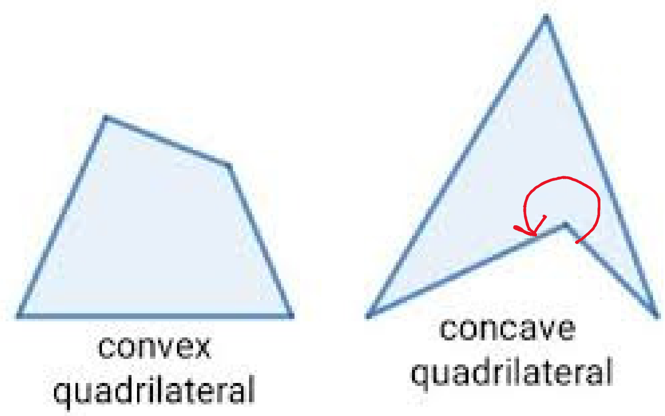
→ Assume it can be done

↳ Chaos ensues

Example 12

An angle is called **reflex** if it exceeds 180°. Prove that no quadrilateral has more than one reflex angle.





Example 12

An angle is called **reflex** if it exceeds 180°. Prove that no quadrilateral has more than one reflex angle.

Assume there exists a quadrilateral with more than one reflex angle

$\therefore$ interior angle sum exceeds $2 \times 180° = 360°$

This contradicts the fact that the interior angle sum of any quadrilateral is exactly 360°

$\therefore$ there can not be more than one reflex angle

Basic outline of a proof by contradiction

1. Assume that the statement we want to prove is false.
2. Show that this assumption leads to mathematical nonsense.
3. Conclude that we were wrong to assume that the statement is false.
4. Conclude that the statement must be true.

Example 13

A **Pythagorean triple** consists of three natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2$$

Show that if (a, b, c) is a Pythagorean triple, then a, b and c cannot all be odd numbers.

Example 13

A **Pythagorean triple** consists of three natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2$$

Show that if (a, b, c) is a Pythagorean triple, then a, b and c cannot all be odd numbers.

Let (a,b,c) be a Pythagorean triple $\Rightarrow a^2+b^2=c^2$

Suppose a, b & c are all odd

$\Rightarrow a^2, b^2$ & c^2 are all odd numbers

$\Rightarrow \underbrace{a^2+b^2}_{\text{sum of 2 odd nos is even}}$ is even and c^2 is odd

Since $a^2+b^2=c^2$, this gives a contradiction

$\therefore$ a, b & c can not all be odd numbers

Possibly the most well-known proof by contradiction is the following.

Theorem

$\sqrt{2}$ is irrational.

Proof

This will be a proof by contradiction. Assume that $\sqrt{2}$ is rational. $\Rightarrow \sqrt{2} = \frac{a}{b}$, a & $b \in \mathbb{Z}$, a & b have no common factors.

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$\Rightarrow (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2$, let $b^2 = k$, $k \in \mathbb{Z}$

$\Rightarrow a^2$ is even

$\Rightarrow a$ is even

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even,then n is even.

a *Write down the contrapositive.*

b Prove the contrapositive.

Solution

a If n is odd, then n^2 is odd.

b Assume that n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring n gives

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore n^2 is odd.

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$\Rightarrow a = 2n$, $n \in \mathbb{Z}$

$\Rightarrow a^2 = (2n)^2 = 2b^2$

$\Rightarrow 4n^2 = 2b^2$

$\Rightarrow b^2 = 2n^2$, Let $n^2 = S$, $S \in \mathbb{Z}$

$\Rightarrow b^2 = 2S$

$\Rightarrow b^2$ is even

$\Rightarrow b$ is even

$\therefore$ Both a and b are even $\Rightarrow a$ & b have common factor 2.

Possibly the most well-known proof by contradiction is the *following*.

$\therefore$ The conclusion contradicts the fact that a & b have no common factors

Theorem

$\sqrt{2}$ is irrational. Negation is false $\xrightarrow{\Delta}$ $\therefore$ $\sqrt{2}$ is irrational

Proof

This will be a proof by contradiction. Assume that $\sqrt{2}$ is rational. $\Rightarrow \sqrt{2} = \frac{a}{b}$, a & $b \in \mathbb{Z}$, $b \neq 0$, a & b have no common factors.

$\Rightarrow (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2$

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