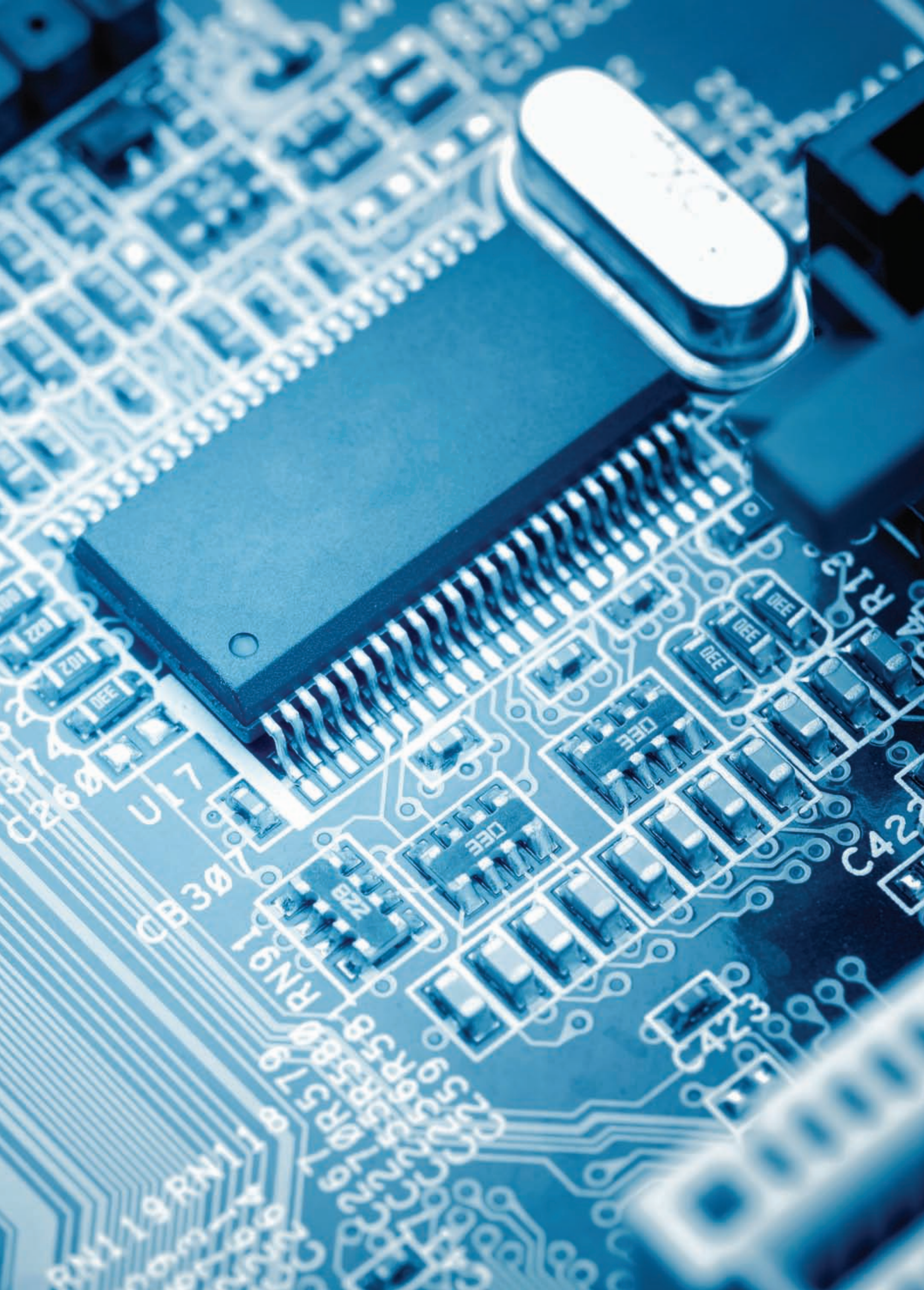

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10

TERMINOLOGY

Argand diagram
closure
complex
complex conjugate
imaginary
inverse
modulus
real
realising the denominator

REAL AND COMPLEX NUMBERS

COMPLEX NUMBERS

Prior learning

COMPLEX NUMBERS

- define the imaginary number i as the root of the equation $x^2 = -1$ (ACMSM067)
- use complex numbers in the form $a + bi$ where a and b are the real and imaginary parts (ACMSM068)
- determine and use complex conjugates (ACMSM069)
- perform complex-number arithmetic: addition, subtraction, multiplication and division. (ACMSM070)

THE COMPLEX PLANE

- consider complex numbers as points in a plane with real and imaginary parts as Cartesian coordinates (ACMSM071)
- examine addition of complex numbers as vector addition in the complex plane (ACMSM072)
- understand and use location of complex conjugates in the complex plane (ACMSM073)

ROOTS OF EQUATIONS

- use the general solution of real quadratic equations (ACMSM074)
- determine complex conjugate solutions of real quadratic equations (ACMSM075)
- determine linear factors of real quadratic polynomials. (ACMSM076)

10.01 IMAGINARY NUMBERS

You have seen before that some quadratic equations do not have a solution in the set of real numbers. For instance, $x^2 = -4$ does not have any real solutions because there is no real number x that gives a negative answer when it is squared. If, however, we make such a number and give it a name, then we can solve these equations and many more.

Unlike real numbers, this number cannot be plotted on the real number line. Although it is imaginary in this sense, the mathematics of i has turned out to be incredibly useful in mathematics, electronics, engineering and technology. Most of modern number theory relies on imaginary numbers for proof. For example, the proof of Fermat's last theorem, that '2 is the only power for which there are distinct positive integer solutions to $a^n + b^n = c^n$', relies heavily on complex and imaginary numbers. The design of electronic circuits would be virtually impossible without imaginary numbers.

> **IMPORTANT**
>
> The **imaginary number** i is defined as $i = \sqrt{-1}$.

Example 1

Use the fact that $i = \sqrt{-1}$ to write the following in terms of i.

a $\sqrt{-4}$ b $\sqrt{-9}$ c $\sqrt{-75}$

Solution

a Write −4 as a product and use the surd rules.

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2 \times i \\ &= 2i\end{aligned}$$

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b Write –9 as a product.

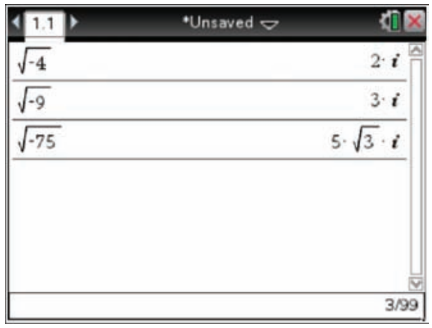
$$\sqrt{-9} = \sqrt{9 \times (-1)}$$
$$= \sqrt{9} \times \sqrt{-1}$$
$$= 3i$$

c Write –75 as a product, then use surd rules to simplify.

$$\sqrt{-75} = \sqrt{25 \times 3 \times (-1)}$$
$$= 5 \times \sqrt{3} \times i$$
$$= 5i\sqrt{3}$$

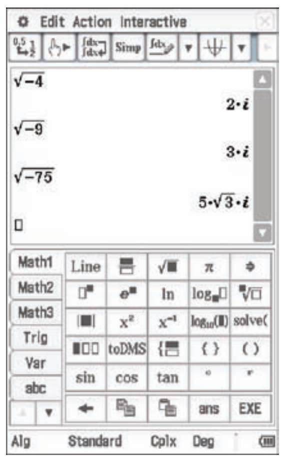
TI-Nspire CAS

Change the Real or Complex setting to Rectangular using doc▾, 7: Settings & Status, 2: comment settings. Then do the calculations shown in the screen shot.



ClassPad

Use the Main $\sqrt[]{\alpha}$ application.
Set the calculator to **Cplx** (complex).
If necessary, tap **Real** at the bottom.



It is much clearer to write $5i\sqrt{3}$ than $5\sqrt{3}i$, because it is clear that the square root applies only to the 3. It is now possible to solve quadratic equations that involve the square root of a negative number.

Example 2

Solve the following equations.

a $x^2 = -16$ b $p^2 = -54$ c $7z^2 + 1 = 0$

Solution

a Take the square root of both sides, noting that there are two possibilities for x.

$$x^2 = -16$$
$$x = \pm\sqrt{-16}$$
$$x = \pm 4i$$

b Simplify the surd.

$$p^2 = -54$$
$$p = \pm\sqrt{9\times 6\times(-1)}$$
$$p = \pm 3i\sqrt{6}$$

c Make z the subject of the equation first, then assign the negative sign to the numerator.

$$7z^2 + 1 = 0$$
$$z^2 = -\frac{1}{7}$$
$$z = \pm\sqrt{\frac{-1}{7}}$$
$$= \pm\frac{i}{\sqrt{7}}$$

Let us now look at powers of the imaginary number i.

Example 3

Evaluate each of the following.

a i^2 **b** i^3 **c** i^4 **d** i^5 **e** i^{50}.

Solution

a Square $\sqrt{-1}$.

$$i^2 = \left(\sqrt{-1}\right)^2$$
$$= -1$$

b Use the fact that $i^2 = -1$.

$$i^3 = i^2 \times i$$
$$= -1 \times i$$
$$= -i$$

c Use the fact that $i^2 = -1$ twice.

$$i^4 = i^2 \times i^2$$
$$= -1 \times (-1)$$
$$= 1$$

d Use the fact that $i^4 = 1$.

$$i^5 = i^4 \times i$$
$$= 1 \times i$$
$$= i$$

e Write the power to get a multiple of 4.
Use the index laws.
Use fact that $i^4 = 1$.

$$i^{50} = i^{48} \times i^2$$
$$= (i^4)^{12} \times (-1)$$
$$= 1^{12} \times (-1)$$
$$= -1$$

IMPORTANT

The powers of i make a pattern that repeats each four numbers: $i, -1, -i, 1, i, -1, -i, 1, \ldots$

EXERCISE 10.01 Imaginary numbers

Concepts and techniques

1 Example 1 Evaluate each of the following using imaginary numbers.

a $\sqrt{-25}$ b $\sqrt{-36}$ c $\sqrt{-20}$ d $\sqrt{-\frac{8}{9}}$

2 Example 2 Solve the following equations using imaginary numbers.

a $x^2 = -1$ b $x^2 = -9$ c $x^2 = -49$ d $x^2 = -\frac{1}{4}$

3 Example 3 Evaluate each power of i.

a i^6 b i^{11} c i^{28} d i^{97}

4 Simplify each expression.

a $i \times (-i)$ b $-i \times (-i)$ c $(-i)^7$ d $5i \times 9i$

5 Expand and simplify the following.

a $(1 + i)(1 - i)$ b $3i(2i + 5)$

6 Evaluate: $1 + i - i^2 + i^3 - i^4 + i^5$

7 By first writing $x^2 + 16 = x^2 - 16i^2$, factorise $x^2 + 16$.

Reasoning and communication

8 Evaluate the series:

a $i + i^2 + i^3 + i^4 + \cdots + i^{100}$ b $\sum_{n=0}^{n=253} i^n$

9 By first writing $1 = i^4$, evaluate each of the following.

a $\frac{1}{i}$ b $\frac{1}{i^3}$ c $\frac{1}{i^4}$ d i^{-5} e i^{-10}

10 Factorise each expression below.

a $x^2 + 4$ b $x^2 + 81$

10.02 COMPLEX NUMBERS

You saw in the previous section that imaginary numbers arise when solving a quadratic equation like $x^2 = -4$. Consider the solution to the equation $x^2 - 3x + 4 = 0$. Using the quadratic formula, we have

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{+3 \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{3 \pm \sqrt{-7}}{2} \end{aligned}$$

$$x = \frac{3 \pm i\sqrt{7}}{2}, \text{ where } i = \sqrt{-1}$$

The solutions can be written in the form $x = \frac{3}{2} + i\frac{\sqrt{7}}{2}$ or $x = \frac{3}{2} - i\frac{\sqrt{7}}{2}$. These solutions are part real and part imaginary. We call these **complex numbers**.

IMPORTANT

A **complex number** is a number that can be written in the form $a + ib$, where a and b are real numbers.

A complex number is often denoted by the letter z, so $z = a + ib$.

The set of complex numbers is denoted by $\boldsymbol{C}$.

The **real part** of $z = a + ib$ is denoted by $\text{Re}(z)$, where $\text{Re}(z) = a$ and the **imaginary part** of $z = a + ib$ is denoted by $\text{Im}(z) = b$.

If $\text{Re}(z) = 0$, then we say that z is *purely imaginary.*

If $\text{Im}(z) = 0$, then we say that z is *purely real* or just *real.*

Example 4

Write down $\text{Re}(z)$ and $\text{Im}(z)$ for each of the following complex numbers.

a $z = 5 - 4i$

b $z = i\sqrt{2} + 9$

c $z = 2x + y + (x - y)i$

d $z = \dfrac{x - iy}{x^2 + y^2}$

Solution

a For $z = 5 - 4i$, the real part is the number 5. The coefficient of i is -4.

$\text{Re}(z) = 5$
$\text{Im}(z) = -4$

b $z = i\sqrt{2} - 9$ is the same as $z = -9 + i\sqrt{2}$.

$\text{Re}(z) = -9$
$\text{Im}(z) = \sqrt{2}$

c For $z = 2x + y + (x - y)i$, all the terms without an i attached form the real part. Those with an i attached form the imaginary part.

$\text{Re}(z) = 2x + y$
$\text{Im}(z) = x - y$

d Write $z = \dfrac{x - iy}{x^2 + y^2}$ as $z = \dfrac{x}{x^2 + y^2} + i\left(\dfrac{-y}{x^2 + y^2}\right)$.

$\text{Re}(z) = \dfrac{x}{x^2 + y^2}$

$\text{Im}(z) = \dfrac{-y}{x^2 + y^2}$

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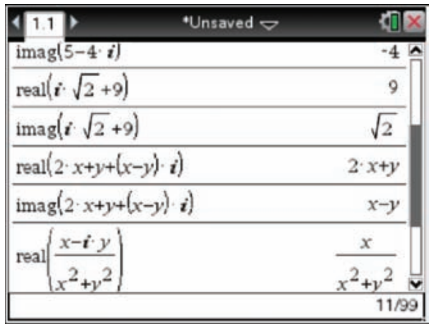
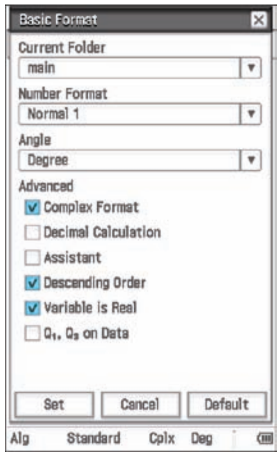
TI-Nspire CAS

Make sure that in Document Settings, Real or Complex is set to Rectangular. Then use menu, 2: Number, 9: Complex Number Tools and 2: Real Part or 3: Imaginary Part to perform the operations for parts **a**, **b**, **c** and **d**.
Use the π▸ button to get i.

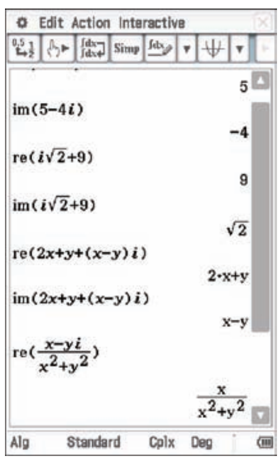


ClassPad

First set up the properties to define variables such as x and y to be real.
Tap ⚙ and then select **Basic Format**.
Make sure that **Variable** is **Real** is ticked.
Complex Format should also be ticked.
Tap Set.



Use the Main $\sqrt{\alpha}$ application.
Set the calculator to **Cplx** (complex) at the bottom, not **Real**.
Use **re** to find the real part and **im** for the imaginary part. (These functions are available on the list obtained by pressing Keyboard then tapping ▾.)
Use i on the Math2 keyboard.



The fact that complex numbers have a real and an imaginary part means that the set of real numbers is a subset of the set of complex numbers ($\boldsymbol{R} \subset \boldsymbol{C}$).

It is now possible to solve any quadratic equation. The solutions could be real, imaginary or complex.

Example 5

Solve each quadratic equation, expressing your answers as complex numbers in the form $a + ib$, where a and b are real.

a $x^2 - 2x + 3 = 0$ **b** $3x^2 + 10x + 15 = 0$

Solution

a Use the quadratic formula.

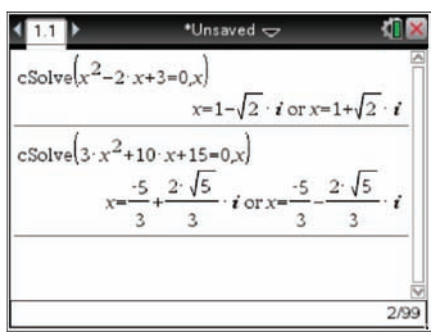
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{+2 \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ x &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2} \end{aligned}$$

b Use the quadratic formula and simplify as far as possible.

$$\begin{aligned} x &= \frac{-10 \pm \sqrt{(10)^2 - 4(3)(15)}}{2(3)} \\ &= \frac{-10 \pm \sqrt{-80}}{6} \\ &= \frac{-10 \pm i4\sqrt{5}}{6} \\ &= \frac{-5 \pm 2i\sqrt{5}}{3} \\ &= -\frac{5}{3} \pm \frac{2\sqrt{5}}{3}i \end{aligned}$$

TI-Nspire CAS

Make sure that Real or Complex is set to Rectangular. Then use cSolve() to find roots that include complex cases.



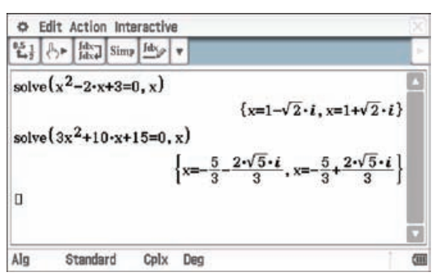
ClassPad

Use the Main $\sqrt{\alpha}$ application.

Set the calculator to **Cplx** (complex).

If necessary, tap **Real** at the bottom.

Use **solve** in the normal manner.



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EXERCISE 10.02 Complex numbers

Concepts and techniques

1 Example 4 Categorise each of the following as real, complex, or purely imaginary.

a $5 + 2i$ b $i\sqrt{5}$ c $2i^2$ d $\frac{4-\sqrt{3}}{2}$

e $\sqrt{-8}$ f $i^2 + i^4$ g $i - \sqrt{2}$ h $\frac{3i - i\sqrt{2}}{i}$

2 Example 4 State Re(z) and Im(z) for each complex number.

a $z = -2 - 4i$ b $z = \frac{7+3i}{4}$

c $z = i\sqrt{2} - 6$ d $\frac{x^2 - iy^2}{x^2 + y^2}$ where x and y are real.

3 Example 4 State the complex number w if Re(w) and Im(w) are the values stated.

a $\text{Re}(w) = 5, \text{Im}(w) = -4$ b $\text{Re}(w) = \sqrt{2}, \text{Im}(w) = \sqrt{7}$

c $\text{Re}(w) = -\frac{1}{2}, \text{Im}(w) = \frac{\sqrt{3}}{2}$ d $\text{Re}(w) = -2\sqrt{2}, \text{Im}(w) = -5\sqrt{2}$

e $\text{Re}(w) = \frac{y}{x^2 + y^2}, \text{Im}(w) = \frac{x}{x^2 + y^2}$ where x and y are real.

4 Example 5 Solve the quadratic equations below, expressing your answers as complex numbers in the form $a + ib$, where a and b are real.

a $x^2 + 49 = 0$ b $z^2 + 1 = 0$ c $w^2 + 25 = 0$

d $x^2 + 4x + 5 = 0$ e $z^2 - z + 2 = 0$ f $w^2 - 2w + 2 = 0$

g $x^2 - 2x + 4 = 0$ h $z^2 + 10 = 6z$ i $w^2 = 4w - 8$

Reasoning and communication

5 Example 4 State Re (z) and Im (z) for each complex number (x and y are real).

a $z = (x + 3) + i(y - 2)$ b $z = x - 3y + (2x + y)i$

c $z = 5i - x + 4 - yi$ d $z = 4x - y - i(3x - 2y)$

e $\frac{x^2 + 2xyi + y^2}{x^2 + y^2}$, where x and y are real.

6 Example 5 Solve each quadratic equation, expressing your answers as complex numbers in the form $a + ib$, where a and b are real.

a $2z^2 + 3z + 4 = 0$ b $3z^2 + 6zi = 2$ c $z^2 + a^2 + b^2 = 2az$

10.03 COMPLEX CONJUGATES

Recall that conjugate surds are surds that make a difference of two squares when multiplied. For instance, the conjugate of $2-\sqrt{5}$ is $2+\sqrt{5}$ and the product $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)$ forms a difference of two squares. Note the answer is *rational*, since $\left(2-\sqrt{5}\right)\left(2+\sqrt{5}\right)=-1$.

> **IMPORTANT**
>
> For a complex number z, where $z = a + ib$ (where a and b are real numbers), the **complex conjugate** of z is denoted by $\overline{z}$ and $\overline{z}=a-ib$.

In the examples above, you may have noticed that the solutions to the quadratic equations are similar to conjugate surds. For instance, the solutions to $x^2 - 2x + 3 = 0$ are $1-i\sqrt{2}$ or $1+i\sqrt{2}$. These are called **complex conjugates**.

In a similar way to surds, the product of complex conjugates forms a difference of two squares. For instance, the conjugate of $1-i\sqrt{2}$ is $1+i\sqrt{2}$ and the product $\left(1-i\sqrt{2}\right)\left(1+i\sqrt{2}\right)$ is a difference of two squares. The answer is *real*, since

$$\begin{aligned}\left(1-i\sqrt{2}\right)\left(1+i\sqrt{2}\right)&=1^2-(i\sqrt{2})^2\\&=1-i^2(\sqrt{2})^2\\&=1-(-1)\times 2\\&=3\end{aligned}$$

> **IMPORTANT**
>
> For a complex number z and its conjugate $\overline{z}$, where $z = a + ib$ (where a and b are real numbers), the product $z\overline{z}$ is real.

Proof

Let $z = a + ib$, then $\overline{z}=a-ib$, where a and b are real. Then

$$\begin{aligned}z\overline{z}&=(a+ib)(a-ib)\\&=a^2-i^2b^2\\&=a^2+b^2\end{aligned}$$

Since a and b are real, then $a^2 + b^2$ is also real.

$\therefore z\overline{z}$ is always real.

Example 6

Write the complex conjugate of each complex number.

a $z = u - vi$ **b** $x + iy - w$ **c** $z = 3 + i(5 - x)$ **d** $\dfrac{ix - y}{x^2 + y^2}$

Solution

a Change the sign of $\text{Im}(z)$. $\bar{z} = u + vi$

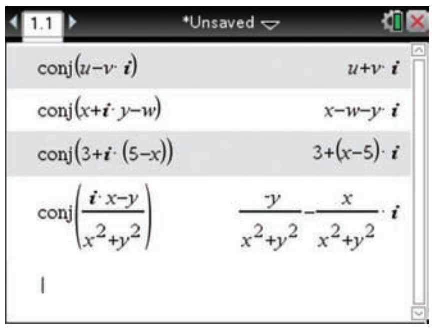
b First group the real parts and imaginary parts separately, i.e. $(x - w) + (iy)$. $x - w - iy$

c Change the sign of $\text{Im}(z)$. $\bar{z} = 3 - i(5 - x)$

d Note that $\dfrac{ix - y}{x^2 + y^2}$ can be written as $\dfrac{-y + ix}{x^2 + y^2}$. $\dfrac{-y - ix}{x^2 + y^2}$

TI-Nspire CAS

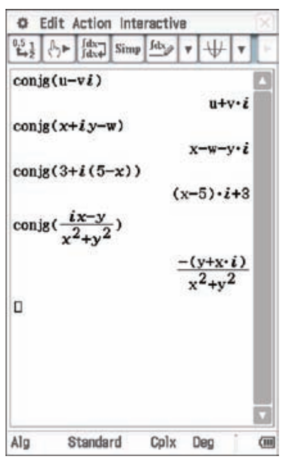
Make sure Real or Complex is set to Rectangular. Then use Complex Conjugate from the Complex Number Tools menu to find the conjugates.



ClassPad

Tap ⚙ then **Basic Format**. Check that this screen has **Complex Format** and **Variable is Real** set, then tap **Set**. (see page 377)

Then use **conjg** to find the conjugates. This only works properly with numbers, but can be used to find the conjugate of a complex expression.



Note that $\dfrac{ix - y}{x^2 + y^2} = \left(\dfrac{-y}{x^2 + y^2}\right) + \left(\dfrac{x}{x^2 + y^2}\right)i$ and $\dfrac{-ix - y}{x^2 + y^2} = \left(\dfrac{-y}{x^2 + y^2}\right) - \left(\dfrac{x}{x^2 + y^2}\right)i$, which shows the conjugates more clearly.

Example 7

Show that $x^2 - 2x + 5 = (x - 1)^2 - 4i^2$ and hence factorise $x^2 - 2x + 5$.

Solution

Start with the RHS. $(x - 1)^2 - 4i^2$

Multiply out the brackets. $= x^2 - 2x + 1 - 4 \times (-1)$

Simplify. $= x^2 - 2x + 1 + 4$

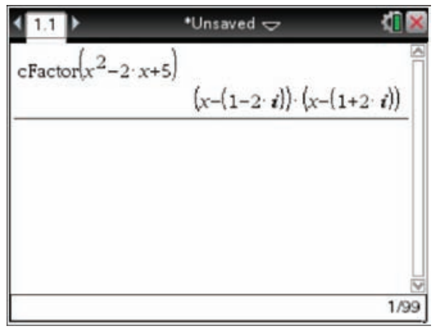
Add the constants. $= x^2 - 2x + 5 = \text{LHS}$

$(x - 1)^2 - 4i^2$ is a difference of two squares. $(x - 1)^2 - 4i^2 = (x - 1 - 2i)(x - 1 + 2i)$

$\therefore\ x^2 - 2x + 5 = (x - 1 - 2i)(x - 1 + 2i)$.

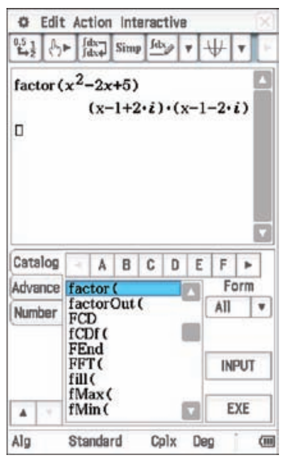
TI-Nspire CAS

Make sure that Real or Complex is set to Rectangular. Then use cFactor() from Complex in the Algebra menu.



ClassPad

Use the $\overset{\text{Main}}{\sqrt{\alpha}}$ application.
Set the calculator to **Cplx** (complex).
Use **factor** in the normal manner.



From Example 7, you could solve $x^2 - 2x + 5 = 0$ using the null factor law to get $x = 1 + 2i$ or $x = 1 - 2i$.

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Example 8

A quadratic equation in the form $x^2 + bx + c = 0$ has complex roots $7 \pm 3i$. Find b and c.

Solution

Write the factorised form of the quadratic equation used to get the roots.

Let the two roots be $\alpha = 7 + 3i$ and $\beta = 7 - 3i$ in the equation $(x - \alpha)(x - \beta) = 0$

Substitute the values of α and β.

$(x - 7 - 3i)\ (x - 7 + 3i) = 0$

Multiply out the brackets.

$x^2 - 7x + 3ix - 7x + 49 - 21i - 3ix + 21i - 9i^2 = 0$

Simplify.

$x^2 - 14x + 49 - 9 \times (-1) = 0$

Add the constants.

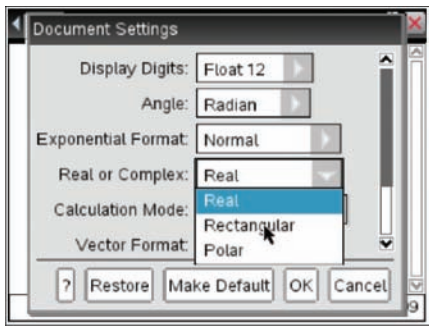
$x^2 - 14x + 58 = 0$

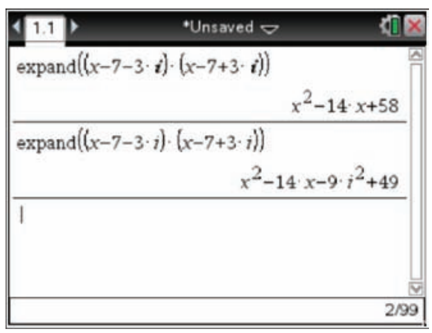
Write the answers.

$b = -14$ and $c = 58$

TI-Nspire CAS

Change your Documents Settings using doc▾, 7: Settings & Status and 2: Document Settings. Change Real or Complex to Rectangular. Then use menu, 3: Algebra and 3: Expand to enter the expression on the LHS of the equation. i is in the π▸ menu.
Note that you must use the ***i*** from the π▸ menu. If you type I from the keyboard it will be treated as a variable.





ClassPad

Use **cExpand** for complex expressions.
Note that you must use the i from the Math2 or Math3 or Trig menu.

EXERCISE 10.03 Complex conjugates

Concepts and techniques

WS Complex conjugates

1 Example 6 State the conjugate $\overline{z}$ for each complex number below.

a $z = -1 - i$

b $z = \dfrac{3+i}{2}$

c $z = i\sqrt{3} - 1$

d $x - y + (2x - 3y)i$, where x and y are real

e $\dfrac{x^2 + iy^2}{x^2 + y^2}$, where x and y are real

2 Show that the answer to each of the following products is real.

a $(2 + i)(2 - i)$

b $\dfrac{5+2i}{2} \times \dfrac{5-2i}{2}$

c $(i\sqrt{5} - 3)\left(-i\sqrt{5} - 3\right)$

3 Example 7 By first writing $x^2 + 2x + 3$ in the form $x^2 + 2x + 1 - 2i^2$, factorise $x^2 + 2x + 3$.

4 Use the technique shown in question **5** to factorise each of the quadratic expressions.

a $x^2 + 4x + 5$

b $x^2 - 6x + 13$

c $x^2 - 2x + 2$

5 Use the quadratic formula to solve the following equations.

a $x^2 + x + 4 = 0$

b $x^2 - 3x + 3 = 0$

c $x^2 + 6x + 10 = 0$

d $2x^2 + x + 4 = 0$

6 Example 8 Find a quadratic equation for which the two given complex numbers are roots.

a $-1 + i, -1 - i$

b $1 - 3i, 1 + 3i$

c $\sqrt{3} - i, \sqrt{3} + i$

Reasoning and communication

7 Given each of the following quadratic equations with real and imaginary coefficients, use the quadratic formula to find the roots.

a $x^2 + ix + 1 = 0$

b $x^2 - 2ix - 1 = 0$

c $ix^2 - x + 2i = 0$

8 a Look at each of the equations in question **5**. Were the coefficients real? Were the solutions to each equation complex conjugate pairs? Complete the statement:
If the coefficients of a quadratic equation are real, then the roots …

b Look at each of the equations in question **7**. Were the coefficients real? Were the solutions to each equation complex conjugate pairs? Complete the statement:
If the coefficients of a quadratic equation are not all real, then the roots …

9 Prove that for any complex number $z = x + yi$

a $z + \overline{z} = 2\text{Re}(z)$

b $z - \overline{z} = 2\text{Im}(z)$

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10.04 OPERATIONS WITH COMPLEX NUMBERS

Just like real numbers, complex numbers can be added, subtracted, multiplied and divided. When dealing with surds, we group or equate the *rational* and *irrational* parts. When dealing with complex numbers, we group or equate the *real* and *imaginary* parts.

IMPORTANT

For complex numbers $a + ib$ and $c + id$ (where a, b, c and d are real numbers), $a + ib = c + id$ if and only if $a = c$ AND $b = d$.

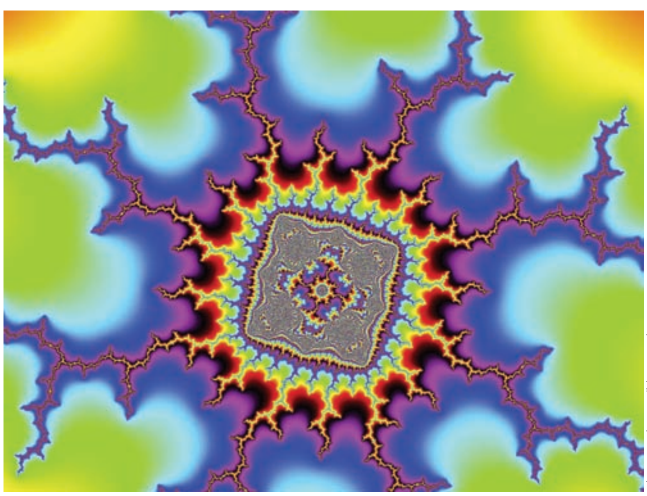
Example 9

Find the value of real numbers x and y if $x - y + i(x + 3y) = 3 + 11i$

Solution

Write the equation.	$x - y + i(x + 3y) = 3 + 11i$
Equate the real parts and the imaginary parts to make two equations in x and y.	$x - y = 3$ [1] $x + 3y = 11$ [2]
Solve simultaneously.	$4y = 8$ [2] − [1] $y = 2$
Substitute in one of the equations.	$x - 2 = 3$ $x = 5$
Write the answers.	$x = 5$ and $y = 2$

You could do Example **9** on your CAS calculator by solving the equation for x and then solving for y, but it is quicker to do it without your calculator.



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Example 10

Simplify each of the following. Express your answer in the form $a + bi$, where a and b are real.

a $(3 + 4i) + (5 - 2i)$

b $(1 - 5i) - (2 - 4i)$

c $(3 + 2i)(4 - 3i)$

d $6(2 + i) - 2i(5 + 7i)$

Solution

a Group the real parts and the imaginary parts.

$$\begin{aligned}(3 + 4i) + (5 - 2i) &= (3 + 5) + i(4 - 2)\\ &= 8 + 2i\end{aligned}$$

b Group the real parts and the imaginary parts.

$$\begin{aligned}(1 - 5i) - (2 - 4i) &= (1 - 2) + i(-5 + 4)\\ &= -1 - i\end{aligned}$$

c Expand like a binomial expansion.
Recall that $i^2 = -1$.

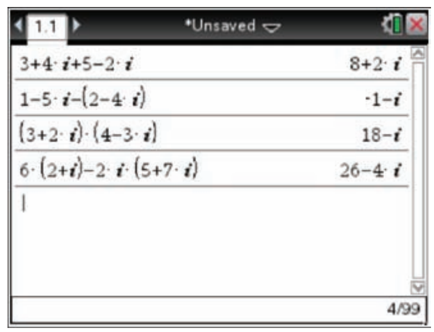
$$\begin{aligned}(3 + 2i)(4 - 3i) &= 12 - 9i + 8i - 6i^2\\ &= 12 - i + 6\\ &= 18 - i\end{aligned}$$

d Remove the brackets and add like terms.

$$\begin{aligned}6(2 + i) - 2i(5 + 7i) &= 12 + 6i - 10i - 14i^2\\ &= 12 - 4i + 14\\ &= 26 - 4i\end{aligned}$$

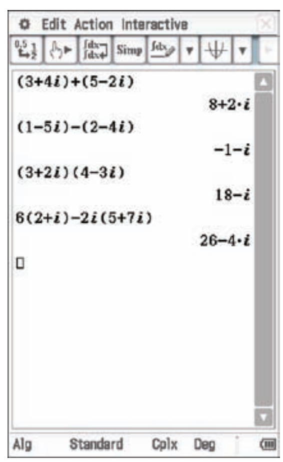
TI-Nspire CAS

You can do these on your calculator by entering the expressions as calculations. Your calculator will automatically express them in the form $a + bi$ when set in Rectangular mode (see page 377).



ClassPad

With numeric expressions, just enter the expression and press EXE.



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When dividing two complex numbers, we multiply by the complex conjugate of the denominator. This is called **realising the denominator**. It is similar to surds, where we rationalise the denominator.

> **IMPORTANT**
>
> To **realise a denominator** of a complex number, you multiply the number by 1 in the form $\frac{\bar{z}}{\bar{z}}$.

Example 11

Simplify by realising the denominator. Give your answer in the form $a + bi$, where a and b are real.

a $\dfrac{3+2i}{4-3i}$ **b** $\dfrac{x-iy}{u+iv}$, where x, y, u and $v \in \boldsymbol{R}$

Solution

a The complex conjugate of $4 - 3i$ is $4 + 3i$.
Recall from the previous section that $(a + bi)(a - bi) = a^2 + b^2$.

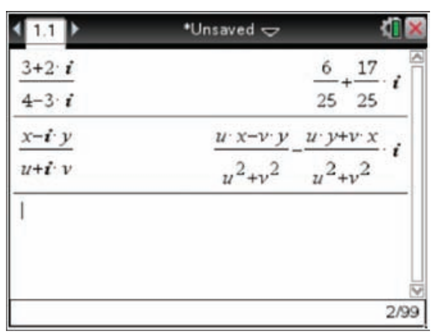
$$\frac{3+2i}{4-3i} \times \frac{4+3i}{4+3i} = \frac{12+9i+8i+6i^2}{4^2+3^2}$$
$$= \frac{12+17i-6}{25}$$
$$= \frac{6+17i}{25}$$
$$= \frac{6}{25} + \frac{17}{25}i$$

b Multiply by the complex conjugate, then group the real and imaginary parts.

$$\frac{x-iy}{u+iv} \times \frac{u-iv}{u-iv} = \frac{xu-xvi-uyi+vyi^2}{u^2+v^2}$$
$$= \frac{(ux-vy)+i(-vx-uy)}{u^2+v^2}$$
$$= \frac{(ux-vy)}{u^2+v^2} + \frac{(-vx-uy)}{u^2+v^2}i$$

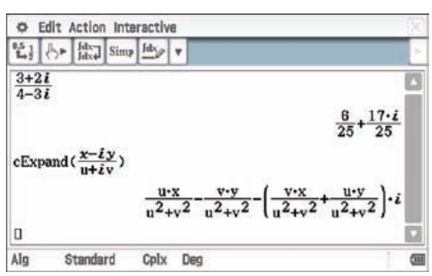
TI-Nspire CAS

Do the calculations in Rectangular mode.



ClassPad

With numeric expressions, just enter the expression and press **EXE**.
With algebraic expressions, use **cExpand**.



EXERCISE 10.04 Operations with complex numbers

WS Complex number operations

Concepts and techniques

1 Example 9 Find the value of a and b in each of the following, where a and b are real.

a $a + 2bi = 7 - 8i$ b $(a + 4i) - (b + bi) = 3 - 2i$

c $2a + b + 3ai - 2bi = 1 + 5i$

2 Example 10 Simplify each of the following expressions, leaving your answer in the form $a + ib$, where a and b are real.

a $(1 - 3i) + (2 + i)$ b $(6 + 5i) - (3 + 4i)$

c $(3 - 2i)(1 + 9i)$ d $2(3 + 2i) + 4i(i - 7)$

3 If $z = 2 + 3i$ and $w = 5 - 2i$, find the following in $a + ib$ form.

a $4z + w$ b $z - w$ c z^2 d zw

e $\overline{z}\,\overline{w}$ f $(z + w)^2$ g $\overline{w}^2$ h $3z - wi$

4 Example 11 Simplify by realising the denominator. Give your answer in the form $a + ib$, where a and b are real.

a $\frac{1}{1+i}$ b $\frac{2-i}{2+i}$ c $\frac{3+4i}{2-3i}$ d $\frac{i}{i-4}$ e $\frac{2-i}{5i}$

5 If $u = -1 + 2i$ and $v = 2 - 3i$, find the following in $a + ib$ form.

a $\frac{1}{u}$ b $\frac{u}{v}$ c $\frac{3}{u^2}$ d $\frac{1}{iv}$ e $\frac{u-1}{v+i}$

6 Simplify each of the following.

a $(2 - i)^2 + 3(1 + 2i)(1 - 2i)$ b $(3 + 5i)(5 - 3i) - (3 - 2i)^2$

c $\frac{1+2i}{2+i} + \frac{1-i}{1-2i}$ d $\frac{1+3i}{(2-i)^2}$

7 Find the value of X and Y where $X, Y \in \boldsymbol{R}$, if $\frac{5-4i}{4+3i} = X + iY$.

Reasoning and communication

8 If $z = x + iy$ where $X, Y \in \boldsymbol{R}$, express the following in the form $X + Yi$, where $X, Y \in \boldsymbol{R}$.

a $\frac{1}{z}$ b $\frac{1}{\overline{z}}$ c $\frac{1}{z^2}$ d $\frac{1}{iz}$ e $\frac{z-i}{z+i}$

9 Show that $z = 1 + i$ is a solution of the equation $z^2 - 2z + 2 = 0$.

10 Show that $z = \frac{-1+i\sqrt{3}}{2}$ is a solution of the equation $z^2 + z + 1 = 0$.

11 Show that $w = \frac{-1-i\sqrt{3}}{2}$ is a solution of the equation $w^3 = 1$.

12 a Show that $\left(x - 1 - i\sqrt{3}\right)\left(x - 1 + i\sqrt{3}\right) = x^2 - 2x + 4$.

b Show that $\left(x + 1 - i\sqrt{3}\right)\left(x - 1 + i\sqrt{3}\right) = x^2 + 2 + 2i\sqrt{3}$.

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10.05 THE COMPLEX PLANE

It is possible to represent the complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) geometrically on a two-dimensional plane similar to the Cartesian plane, with a horizontal axis denoted by x or Re(z) and vertical axis denoted by y or Im(z). This plane is called an **Argand diagram** or **Argand plane**, after the French mathematician Jean-Robert Argand (1768–1822).

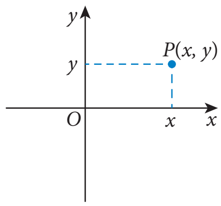


Figure 1

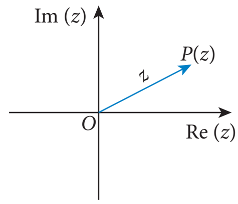


Figure 2

The complex number $z = x + yi$ can be represented by the point $P(x, y)$ (Figure 1) or by the vector $\overrightarrow{OP}$ (Figure 2).

Real numbers are plotted along the x-axis so it is labelled Re(z), Re or sometimes x.

Purely imaginary numbers are plotted along the y-axis so it is labelled Im(z), Im, iy or y.

[By convention, the complex number $z = 0 + 0i$ is just written as 0 on the Argand diagram. It is the origin.]

All complex numbers $z = a + bi$ (where $a \neq 0$, $b \neq 0$) correspond to a unique point (a, b) or vector on the Argand diagram.

Example 12

Represent each of the following numbers as a point on an Argand diagram.

a $u = 2 + 3i$ **b** $w = -1 - 4i$ **c** $v = (1 + 2i)(1 - 2i)$ **d** $z = \dfrac{3}{i}$

Solution

a Write u as a point. $u = 2 + 3i$ is shown as $(2, 3)$ or $2 + 3i$

b Write w as a point. $w = -1 - 4i$ is shown as $(-1, -4)$ or $-1 - 4i$

c Change v to the form $a + bi$ first.

$$\begin{aligned} v &= (1 + 2i)(1 - 2i) \\ &= 1 - 2i + 2i - 4i^2 \\ &= 1 + 4 \\ &= 5 + 0i \end{aligned}$$

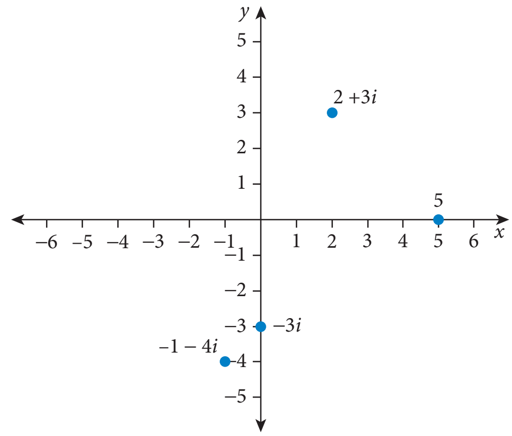
$v = (1 + 2i)(1 - 2i)$ is shown as $(5, 0)$ or 5.

d Change z to the form $a + bi$ first.

$z = \frac{3}{i} = \frac{3}{i} \times \frac{i}{i} = \frac{3i}{-1} = -3i = 0 - 3i$

z is shown as $(0, -3)$ or $-3i$.

Show the numbers on the diagram.



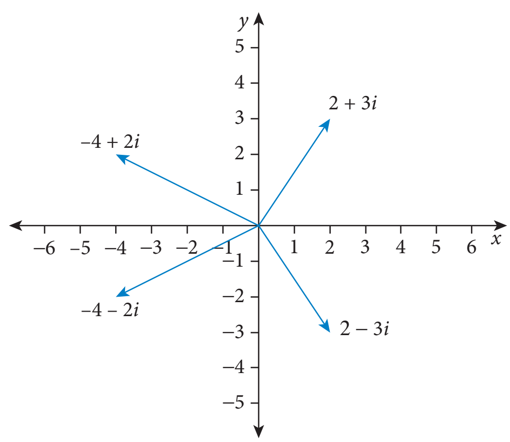
Example 13

Given that $z = 2 - 3i$ and $w = -4 + 2i$, show $z, \overline{z}, w, \overline{w}$ as vectors on an Argand diagram. What do you notice about the geometric relationship between complex conjugates?

Solution

Write the conjugates. $\overline{z} = 2 + 3i$ and $\overline{w} = -4 - 2i$

Plot $z = 2 - 3i$, $w = -4 + 2i$, $\overline{z} = 2 + 3i$ and $\overline{w} = -4 - 2i$ with arrows on the ends to show they are vectors.



Comment on the conjugates. The conjugates are reflections in the x-axis.

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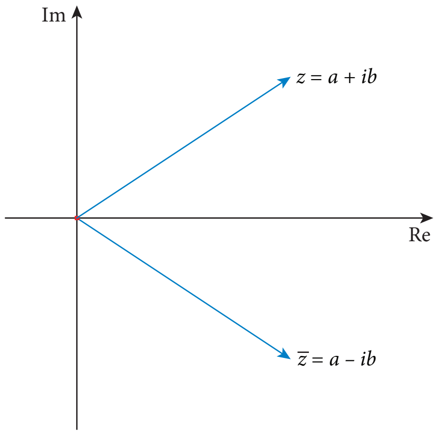
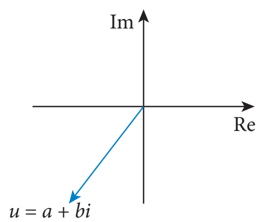


IMPORTANT

On an Argand diagram, if $z = a + bi$, then the complex conjugate $\overline{z} = a - bi$ is the reflection of the vector $z = a + bi$ in the x-axis.

Example 14

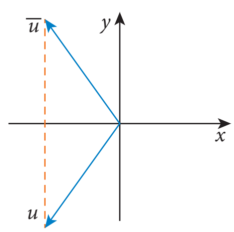
The vector $u = a + bi$ is plotted below on an Argand diagram.



Plot the conjugate vector $\overline{u}$.

Solution

The conjugate is the reflection across the x-axis.



EXERCISE 10.05 The complex plane

Concepts and techniques

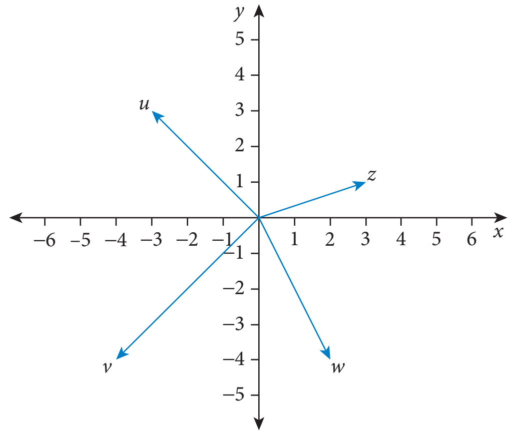
1 Example 12 Given the complex numbers below,

i state the points corresponding to each complex number

ii plot the points corresponding to each complex number on an Argand diagram.

a $a = 1 + 2i$ **b** $b = 3 - i$ **c** $c = -3 - 4i$ **d** $d = -2 + 2i$

2 State the complex number represented by each vector on the Argand diagram below.



3 Example 13 On an Argand diagram, plot the vectors corresponding to each complex number z with its conjugate $\bar{z}$.

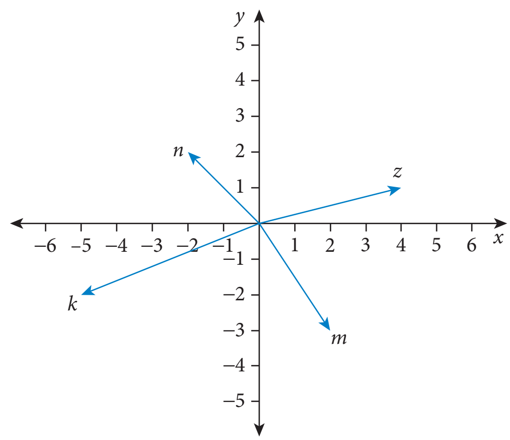
a $z=\sqrt{3}-i$ b $z=\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}$ c $z=-3-2i$

4 Example 14 The vectors corresponding to four complex numbers are shown in the Argand diagram on the right.

a State the complex number representing each vector z, n, k and m.

b State the complex conjugate of each.

c Plot $\bar{z}, \bar{n}, \bar{k}$ and $\bar{m}$ on an Argand diagram.



5 Represent each of the following numbers as points on an Argand plane.

a $z=(1+i)(2-i)$ b $w=\dfrac{1}{-1+i}$ c $v=\left(\sqrt{2}-i\right)^2$

6 If $z=3+2i$,

a calculate the complex numbers

i $w=iz$ ii $v=i^2z$ iii $u=i^3z$

b plot the corresponding vectors z, w, v and u.

c How is the position of a vector affected if you multiply a complex number by i ?

Reasoning and communication

7 If $z+\bar{z}=0$, explain why z lies on the imaginary axis.

8 If z is a vector on the complex plane, explain why $z+\bar{z}$ lies on the real axis, if $z+\bar{z}\neq 0$.

9 If $z=\bar{z}$, explain the position of z on the Argand diagram.

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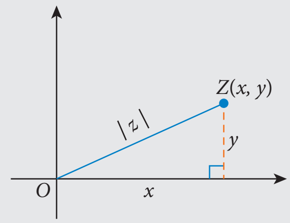
10.06 THE MODULUS OF A COMPLEX NUMBER

For a complex number $z = x + yi$, represented by the point $Z(x, y)$, the length of the vector $\overrightarrow{OZ}$ can be found using Pythagoras' theorem. It is called the **modulus** of z and is denoted by $|z|$. Since it is a length, $|z| \geq 0$.

IMPORTANT

The length of the vector $z = x + yi$ is called the **modulus**, written as **mod** z or $|z|$.

$$|z| = \sqrt{x^2 + y^2} \text{ or } |z| = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}$$



Example 15

Find $|z|$ for each of the following complex numbers.

a $z = 1 - i\sqrt{3}$ **b** $z = -2 + 2i$ **c** $z = \dfrac{1+i}{\sqrt{2}}$

Solution

a Identify the real and imaginary parts. Do not include i.

$$|z| = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{4} = 2$$

b Use the formula.

$$|z| = \sqrt{(-2)^2 + (2)^2}$$
$$= \sqrt{8}$$

Simplify the surd.

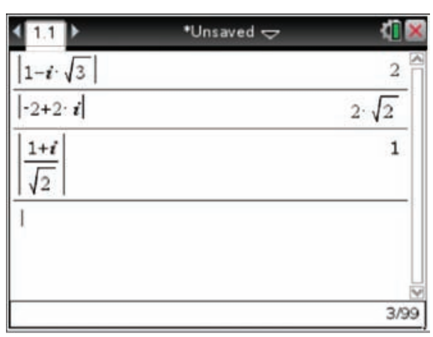
$$= 2\sqrt{2}$$

c Separate into real and imaginary parts.

$$|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$
$$= \sqrt{\frac{1}{2} + \frac{1}{2}}$$
$$= \sqrt{1} = 1$$

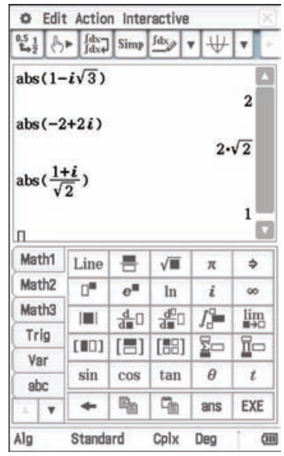
TI-Nspire CAS

Make sure that Real or Complex is set to Rectangular. Then use Magnitude from the Complex Number Tools.



ClassPad

To find the modulus of a complex number, use the function **abs**.
This is because the modulus of a complex number is an extension of the idea of absolute value for real numbers.



Example 16

Evaluating the modulus of a complex number 1

Evaluating the modulus of a complex number 2

Find in simplest form.

a $\left|x-y+i(x+y)\right|$ b $\left|\dfrac{x-yi}{x^2+y^2}\right|$

Solution

a Identify the real and imaginary parts.

$\text{Re}[x-y+i(x+y)] = x-y$
$\text{Im}[x-y+i(x+y)] = x+y$

Use the formula.

$$\left|x-y+i(x+y)\right| = \sqrt{(x-y)^2+(x+y)^2}$$
$$= \sqrt{2x^2+2y^2}$$

b Use the formula.

$$\left|\frac{x-yi}{x^2+y^2}\right| = \sqrt{\left(\frac{x}{x^2+y^2}\right)^2+\left(\frac{-y}{x^2+y^2}\right)^2}$$

Simplify.

$$= \sqrt{\frac{x^2}{(x^2+y^2)^2}+\frac{y^2}{(x^2+y^2)^2}}$$
$$= \sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}}$$

Cancel and rationalise the surd.

$$= \sqrt{\frac{1}{x^2+y^2}}$$
$$= \frac{1}{\sqrt{x^2+y^2}} \text{ or } \frac{\sqrt{x^2+y^2}}{x^2+y^2}$$

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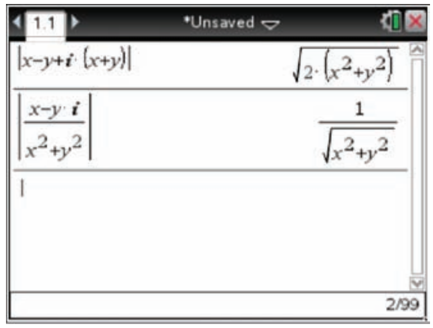
TI-Nspire CAS

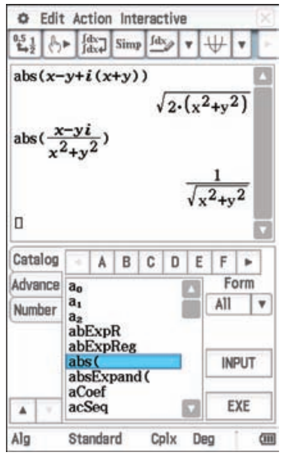
Make sure that Real or Complex is set to Rectangular. You can use Magnitude from the Complex Number Tools with variables.



ClassPad

The function **abs** can also be used with variables, like x and y.

Make sure **Basic Format** (under ⚙) is set to **Variable is Real**. See page 377.



EXERCISE 10.06 The modulus of a complex number

Concepts and techniques

1 Example 15 Find $|z|$ for each of the following complex numbers.

a $z = 3 + 5i$ b $z = -2 + i$ c $z = \sqrt{3} - i$

d $z = 1 + i\sqrt{2}$ e $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ f $z = \frac{3 - 4i}{5}$

2 If $u = 3 - 2i$, find the value of each of the following.

a $|u|$ b $|u^{-1}|$ c $\frac{1}{|u|}$ d $|u^2|$

3 Evaluate the following.

a $|\sqrt{3} + i|$ b $|\sqrt{2} - i\sqrt{7}|$ c $\frac{1}{|6 + 8i|}$ d $\left|\frac{1 - i}{\sqrt{2}}\right|$

4 By first expressing each complex number in the form $a + ib$, find the modulus.

a $5 - i(2 + 3i)$ b $(3 - 2i)(1 + 4i)$ c $2\sqrt{3}(2 + i)$

d $\frac{1}{1 + i\sqrt{3}}$ e $\frac{1 - i}{1 + i}$ f $i\left(\sqrt{2} - i\right)^2$

5 If $z = -2 + 3i$ and $w = 1 - 4i$, show that each of the following is true.

a $|zw| = |z||w|$

b $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

c $|z + w| \leq |z| + |w|$

d $|z - w| \geq |z| - |w|$

6 Example 16 Find in terms of x and y:

a $|x + iy|$

b $\left|\frac{1}{x - iy}\right|$

c $\left|\frac{x + 1 - iy}{x + 1 + iy}\right|$

Reasoning and communication

7 If $z = a + ib$, show that $|z| = |\bar{z}|$. Can you explain geometrically why this is true?

8 If $z = x + yi$, show that $|z|^2 = |z^2|$.

9 If $z = x + yi$, show that $|z|^2 = z\bar{z}$.

10 If $z = x + yi$ and $w = u + vi$, where $u, v, x, y \in \boldsymbol{R}$, prove that

a $|zw| = |z||w|$

b $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

10.07 OPERATIONS IN THE ARGAND PLANE

As discussed in Chapter 3, two vectors can be added or subtracted using the parallelogram rule. Complex numbers can be represented as vectors on the Argand diagram. Using vector geometry, we can add and subtract complex numbers on the Argand diagram.

Example 17

If $z = 1 + 3i$ and $w = 4 + 2i$,

a find $z + w$

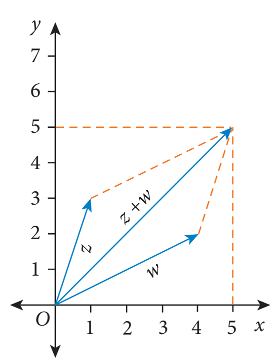
b plot z, w and $z + w$ as vectors on the complex plane.

Solution

a Group the real and imaginary parts.

$$\begin{aligned} z + w &= (1 + 4) + (3 + 2)i \\ &= 5 + 5i \end{aligned}$$

b The vectors z and w form the sides of the parallelogram and $z + w$ is the diagonal.



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Recall that the vector −**z** has the same magnitude as the vector **z** but it is in the opposite direction.

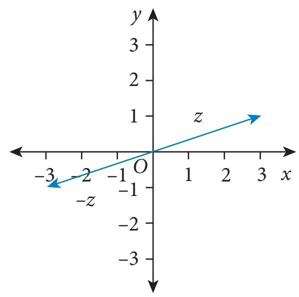
Example 18

Given $z = 3 + i$, plot the following as vectors.

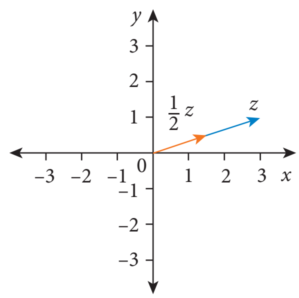
a $-z$ **b** $\frac{1}{2}z$

Solution

a The vector $-z = -3 - i$. It is in the opposite direction.



b $\frac{1}{2}z$ is half the length of z.



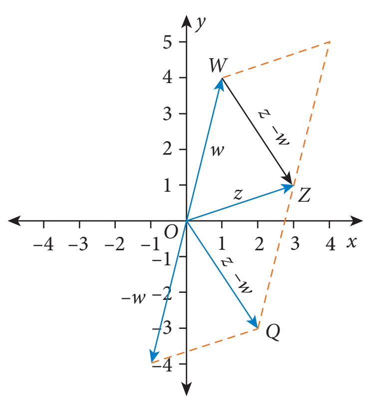
It is possible to subtract two vectors **z** and **w** using −**z** or −**w**, or by using the other diagonal in the parallelogram rule.

Example 19

Given **z** = 3 + i and **w** = 1 + 4i, plot the vector **z** − **w** and label the corresponding point Q.

Solution

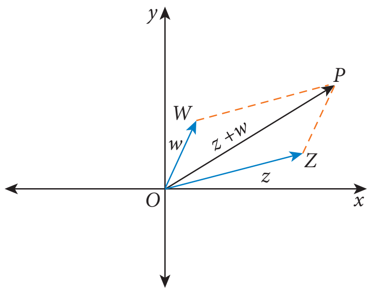
First write **z** − **w** as **z** + (−**w**).
Use the parallelogram rule with **z** and −**w**.
The vector **z** − **w** is the diagonal vector from O. The point $Q(2, -3)$ corresponds with the arrowhead.
Alternatively, use the parallelogram rule with **z** and **w**.
The vector joining Z and W with the arrowhead on Z is also the vector **z** − **w**.

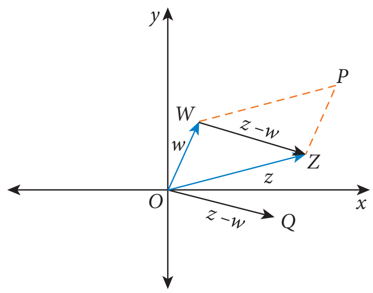


The example above highlights this important point: the vector is movable but the corresponding point is fixed. The point corresponds to the arrowhead of the vector with its tail at the origin because *the vector represents a complex number*.

IMPORTANT

Given the vectors **z** and **w** with corresponding points Z and W respectively, the vectors **z** + **w** and **z** – **w** are the diagonal vectors in the parallelogram $OWPZ$.





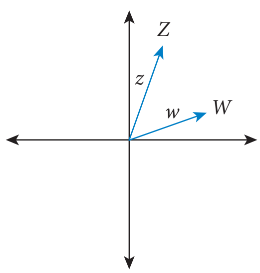
The point P corresponds to the vector **z** + **w**.

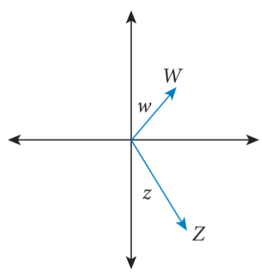
The point Q corresponds to the vector **z** – **w**.

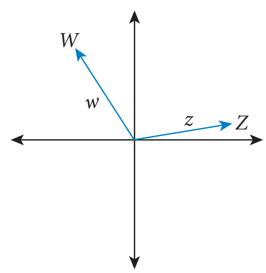
EXERCISE 10.07 Operations in the Argand plane

Concepts and techniques

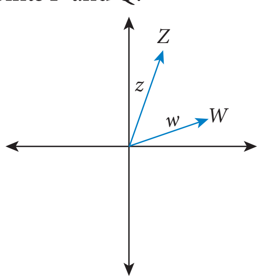
1 Example 17 Copy the diagrams below and use the parallelogram rule to find the vector **z** + **w**. Label the corresponding point P.

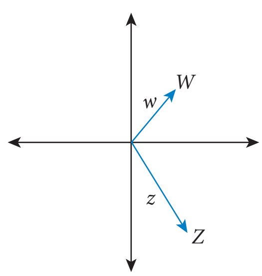
a


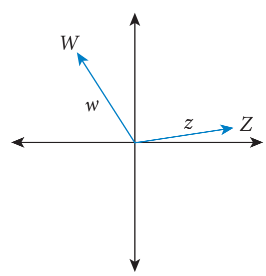
b


c


2 Example 18 Copy the diagrams below and find the vectors –**z** and –**w**. Label the corresponding points P and Q.

a


b


c


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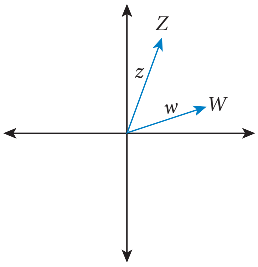
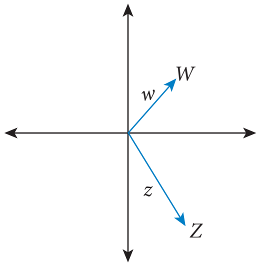
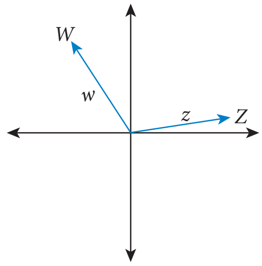
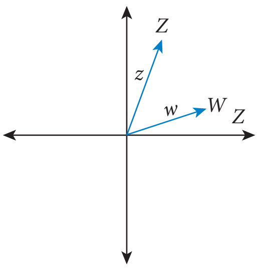
3 Copy the diagrams below and find the vectors $\frac{1}{2}\mathbf{z}$ and $\frac{1}{2}\mathbf{w}$. Label the corresponding points U and V.

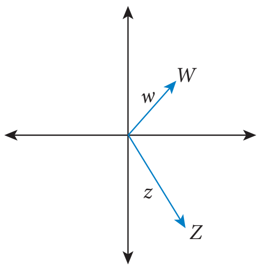
a


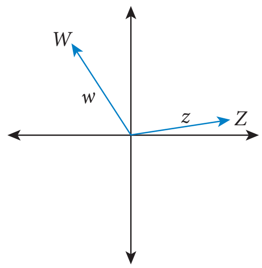
b


c


4 Example 19 Copy the diagrams below and use the parallelogram rule to find the vector $\mathbf{z} - \mathbf{w}$. Label the corresponding point R.

a


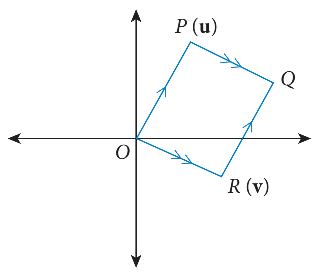
b


c


5 Consider the parallelogram $OPQR$ in the Argand diagram on the right.
Let $\overrightarrow{OP} = \mathbf{u}$ and $\overrightarrow{OR} = \mathbf{v}$.
In terms of $\mathbf{u}$ and $\mathbf{v}$ find expressions for

a $\overrightarrow{OQ}$ b $\overrightarrow{PR}$ c $\overrightarrow{QO}$

d $\overrightarrow{RP}$ e M, the midpoint of OQ.



6 Consider the points M and N with corresponding vectors $\mathbf{m} = -2 + i$ and $\mathbf{n} = 3 + 2i$. The points O, M, L, N form a parallelogram $OMLN$. Find the complex number corresponding to the vertex L.

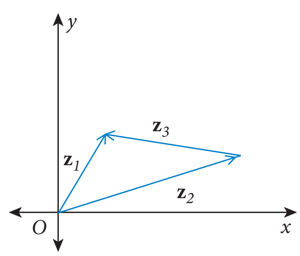
Reasoning and communication

7 Consider the vectors $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$ shown in the diagram. Express

a $\mathbf{z}_3$ in terms of $\mathbf{z}_1$ and $\mathbf{z}_2$

b $\mathbf{z}_1$ in terms of $\mathbf{z}_2$ and $\mathbf{z}_3$

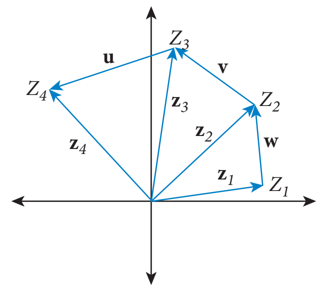
c $\mathbf{z}_2$ in terms of $\mathbf{z}_1$ and $\mathbf{z}_3$



8 Consider any two vectors w_1 and w_2 on an Argand Diagram. Using a diagram, explain why

a $|w_1 + w_2| \le |w_1| + |w_2|$ b $|w_1 - w_2| \ge |w_1| - |w_2|$

9 Consider the points Z_1, Z_2, Z_3, Z_4 corresponding to the complex numbers with vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ and $\mathbf{z}_4$ as shown in the diagram. Show that $\mathbf{z}_4 - \mathbf{z}_1 = \mathbf{w} + \mathbf{u} + \mathbf{v}$.



10.08 PROPERTIES OF COMPLEX NUMBERS

The complex numbers form a field. The complex numbers complete the development of the number system. They are sufficient for performing all the calculations that we need to be able to do, including in complex problems such as the fluid dynamics used in aerospace engineering and design.

IMPORTANT

For any complex numbers, z, w and $v \in \boldsymbol{C}$, the following properties hold.

1. Closure under addition: $z + w \in \boldsymbol{C}$
2. Closure under multiplication: $zw \in \boldsymbol{C}$
3. The commutative law of addition: $z + w = w + z$
4. The commutative law of multiplication: $zw = wz$
5. The associative law of addition: $(z + w) + v = z + (w + v)$
6. The associative law of multiplication: $(zw)v = z(wv)$
7. The distributive law of multiplication over addition: $z(w + v) = zw + zv$
8. Additive identity: $0 + z = z + 0 = z, \forall z \in \boldsymbol{C}$
9. Multiplicative identity: $1 \times z = z \times 1 = z$.
10. Additive inverse: $\forall z \in \boldsymbol{C}\ \exists\ (-z) \in \boldsymbol{C}$ such that $z + (-z) = (-z) + z = 0$
11. Multiplicative inverse: $\forall z \neq 0 \in \boldsymbol{C}\ \exists\ z^{-1} \in \boldsymbol{C}$ such that $zz^{-1} = z^{-1}z = 1$

Example 20

Show that there exists a multiplicative inverse for nonzero numbers in the complex plane.

Solution

We must show that there exists a number z^{-1} that can be written in the form $A + Bi$ such that $z \times z^{-1} = z^{-1} \times z = 1$.	Let $z = a + bi$ The inverse is the reciprocal, $z^{-1} = \dfrac{1}{z}$.

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Write the reciprocal.

$$z^{-1} = \frac{1}{a+bi}$$

Rationalise the denominator.

$$= \frac{1}{a+bi} \times \frac{a-bi}{a-bi}$$
$$= \frac{a-bi}{a^2+b^2}$$
$$= \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

Check the product zz^{-1}.

$$z \times z^{-1} = (a+ib) \times \left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i \right)$$

Multiply out the brackets.

$$= \frac{aa}{a^2+b^2} + \frac{-ab}{a^2+b^2}i + \frac{aib}{a^2+b^2} + \frac{-bb}{a^2+b^2}i^2$$

Simplify.

$$= \frac{a^2}{a^2+b^2} + \frac{-b^2i^2}{a^2+b^2}$$

Use $i^2 = -1$ and put on a common denominator.

$$= \frac{a^2+b^2}{a^2+b^2} = 1$$

Check the product $z^{-1}z$.

$$z^{-1} \times z = \left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i \right) \times (a+ib)$$

Multiply out the brackets.

$$= \frac{aa}{a^2+b^2} + \frac{-ab}{a^2+b^2}i + \frac{aib}{a^2+b^2} + \frac{-bb}{a^2+b^2}i^2$$

Simplify.

$$= \frac{a^2}{a^2+b^2} + \frac{-b^2i^2}{a^2+b^2}$$

Use $i^2 = -1$ and put on a common denominator.

$$= \frac{a^2+b^2}{a^2+b^2}$$
$$= 1$$

Write the result.

For any number $z \neq 0 \in \mathbf{C}$ there exists $z^{-1} = \frac{1}{z}$ such that $z \times z^{-1} = z^{-1} \times z = 1$. **QED**

There are also other important properties of complex numbers. These are stated below.

IMPORTANT

Properties of complex numbers

1 $|z|^2 = z\bar{z}$

2 $|z|^n = |z^n|$

3 $|z_1z_2| = |z_1||z_2|$

4 $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

5 $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

6 $\overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$

INVESTIGATION Complex numbers in real life

As mentioned earlier in the chapter, many of the electronic devices you use today such as computers, Smartphones and MP3 players rely on the mathematics of complex numbers. Investigate the following milestones in mathematical and scientific history.

1 Who was the first person to use the idea of imaginary numbers?

2 Who first used the notation $i = \sqrt{-1}$?

3 When was the first electronic device invented? By whom?

4 Give an example of how complex numbers can be utilised in electronics.

Example 21

Prove that for two complex numbers z_1 and z_2 that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.

Solution

Express z_1 and z_2 in Cartesian form.

Let $z_1 = a + bi$ and $z_2 = c + di$.

Consider the LHS.

$$\text{LHS} = \left|\frac{z_1}{z_2}\right|$$

Realise the denominator.

$$= \left|\frac{a+bi}{c+di}\right|$$

$$= \left|\frac{a+bi}{c+di} \times \frac{c-di}{c-di}\right|$$

Group the real and imaginary parts and use the formula for the modulus.

$$= \left|\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}\right|$$

$$= \sqrt{\frac{(ac+bd)^2}{(c^2+d^2)^2} + \frac{(bc-ad)^2}{(c^2+d^2)^2}}$$

$$= \frac{\sqrt{a^2c^2 + 2abcd + b^2d^2 + b^2c^2 - 2abcd + a^2d^2}}{c^2+d^2}$$

Re-factorise.

$$= \frac{\sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)}}{c^2+d^2}$$

$$= \frac{\sqrt{(c^2+d^2)(a^2+b^2)}}{c^2+d^2}$$

Cancel.

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

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Consider the RHS.	$\text{RHS} = \dfrac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
	$= \dfrac{\lvert a + bi \rvert}{\lvert c + di \rvert}$
Find the modulus of the numerator and denominator.	$= \dfrac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}}$
	$= \sqrt{\dfrac{a^2 + b^2}{c^2 + d^2}}$
Write the conclusion.	$\text{LHS} = \text{RHS}$
	$\therefore \left\lvert \dfrac{z_1}{z_2} \right\rvert = \dfrac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$ **QED**

EXERCISE 10.08 Properties of complex numbers

Concepts and techniques

WS Complex conjugates and inverses

1 Example 20 Let $z_1 = a + bi$, $z_2 = c + di$ and $z_3 = e + fi$. Prove the properties below for the set of complex numbers $\boldsymbol{C}$.
   a The closure under addition
   b Closure under multiplication
   c The commutative law of addition
   d The commutative law of multiplication
   e The associative law of addition
   f The associative law of multiplication
   g The distributive law of multiplication over addition

2 Let $z = x + yi$, $1 = 1 + 0i$ and $0 = 0 + 0i$. Write down the
   a additive inverse of $z = x + yi$
   b multiplicative inverse of $z = x + yi$
   c the additive identity for $z = x + yi$
   d the multiplicative identity for $z = x + yi$.

3 Example 21 Show that each of the following is true for $z = 3 - 4i$.
   a $\lvert z \rvert^2 = z\bar{z}$
   b $\lvert z \rvert^2 = \lvert z^2 \rvert$

4 Let $z_1 = 1 - 2i$ and $z_2 = -1 + i$. Show that each property is true.
   a $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$
   b $\left\lvert \dfrac{z_1}{z_2} \right\rvert = \dfrac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$
   c $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
   d $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

Reasoning and communication

5 Prove the following properties $\forall z \in \boldsymbol{C}$.

a $|z|^2 = z\bar{z}$

b $|z|^2 = |z^2|$

c $|z_1 z_2| = |z_1||z_2|$

d $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

e $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

f $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$

6 Show that $z + \bar{z}$ is real $\forall z \in \boldsymbol{C}$.

7 Show that $z\bar{z}$ is real $\forall z \in \boldsymbol{C}$.

8 Is it true that $\frac{1}{|z|^2} = \frac{1}{|\bar{z}|^2}$ $\forall z \in \boldsymbol{C}$? Give reasons for your answer.

10.09 QUADRATIC EQUATIONS

You have seen before that a quadratic equation written in the form $ax^2 + bx + c = 0$, where a, b and c are real, has two roots that can be real, imaginary or complex. If the two roots are α and β, then we can write

$$\begin{aligned} ax^2 + bx + c &\equiv a(x - \alpha)(x - \beta) \\ &\equiv a(x^2 - \alpha x - \beta x + \alpha\beta) \\ &\equiv a[x^2 - (\alpha + \beta)x + \alpha\beta]. \end{aligned}$$

You can both solve a quadratic equation *and* create a quadratic equation or expression given the roots.

Example 22

Solve each quadratic equation in the complex plane and classify the roots as real, imaginary or complex.

a $x^2 - 6x - 7 = 0$

b $x^2 - 6x + 7 = 0$

c $x^2 - 6x + 10 = 0$

Solution

a Factorise to solve.

$x^2 - 6x - 7 = 0$

$(x - 7)(x + 1) = 0$

$x = 7$ or 1

The roots are real.

b $x^2 - 6x + 7 = 0$ does not factorise.
Use the quadratic formula.

$$x^2 - 6x + 7 = 0$$

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} \\ &= \frac{6 \pm \sqrt{8}}{2} \\ &= \frac{6 \pm 2\sqrt{2}}{2} \\ &= 3 \pm \sqrt{2} \end{aligned}$$

The roots are real.

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c $x^2 - 6x + 10 = 0$ does not factorise. Use the quadratic formula OR create a difference of two squares.

$$x^2 - 6x + 10 = 0$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-4}}{2}$$

$$= \frac{6 \pm 2i}{2}$$

$$= 3 \pm i$$

OR

$x^2 - 6x + 10 = 0$

$x^2 - 6x + 9 - i^2 = 0$

$(x - 3)^2 - i^2 = 0$

$(x - 3 - i)(x - 3 + i) = 0$

$x - 3 - i = 0$ or $x - 3 + i = 0$

$x = 3 + i$ or $x = 3 - i$

Note that the roots are complex conjugates.

$\therefore\ x = 3 \pm i$

The roots are complex and form a conjugate pair.

IMPORTANT

Theorem:

If a quadratic equation $ax^2 + bx + c = 0$, has real coefficients a, b and c, then, if $\alpha = n + mi$ is a root (n and m are real), then its conjugate $\overline{\alpha} = n - mi$ will also be a root.

The proof of this theorem relies on the four properties of conjugates. Consider any two complex numbers $z_1 = A + Bi$ and $z_2 = C + Di$, where $A, B, C, D \in \boldsymbol{R}$.

1 If $z = 0$, then $\overline{z} = 0$.

2 If $\overline{z_1 + z_2} = 0$, then $\overline{z_1} + \overline{z_2} = 0$.

3 If $\overline{kz} = 0$, then $k\overline{z} = 0$, where $k \in \boldsymbol{R}$.

4 If $\overline{z_1 z_2} = 0$, then $\overline{z_1}\,\overline{z_2} = 0$.

Now we are able to prove the theorem.

Proof

If $\alpha = n + mi$ is a root of $ax^2 + bx + c = 0$, then $\alpha = n + mi$ must satisfy $ax^2 + bx + c = 0$.

That is, $a(\alpha)^2 + b(\alpha) + c = 0$.

Now we must prove that $\overline{\alpha} = n - mi$ must satisfy $ax^2 + bx + c = 0$. Consider

$$a(\alpha)^2 + b(\alpha) + c = 0$$

$$\overline{a(\alpha)^2 + b(\alpha) + c} = 0$$

$$\overline{a(\alpha)^2} + \overline{b(\alpha)} + c = 0$$

$$a\overline{(\alpha)^2} + b\overline{(\alpha)} + c = 0$$

$$a(\overline{\alpha})^2 + b(\overline{\alpha}) + c = 0$$

$\therefore$ both α and $\overline{\alpha}$ satisfy $ax^2 + bx + c = 0$.

This means that complex roots of quadratic equations with *real* coefficients always come in conjugate pairs.

IMPORTANT

If a quadratic equation $ax^2 + bx + c = 0$ has real coefficients a, b and c, then it is called a *real* quadratic equation. The corresponding expression $ax^2 + bx + c$ is called a *real* quadratic expression.

Example 23

Show that $3 - i\sqrt{2}$ is a root of the equation $z^2 - 6z + 11 = 0$ and find the other root.

Solution

Substitute $3 - i\sqrt{2}$ into the RHS.

$$\left(3 - i\sqrt{2}\right)^2 - 6\left(3 - i\sqrt{2}\right) + 11$$

Simplify.

$$= 9 - 6i\sqrt{2} - 2 - 18 + 6i\sqrt{2} + 11$$

$$= 0$$

Write the conclusion.

$\therefore\ 3 - i\sqrt{2}$ is a root.

Complex roots of real quadratic equations come in conjugate pairs.

The other root is $3 + i\sqrt{2}$.

It is easy to form the real quadratic equation or expression whose roots or zeros are α and $\overline{\alpha}$.

IMPORTANT

If a real quadratic equation $ax^2 + bx + c = 0$, or the corresponding real expression $ax^2 + bx + c$ has complex conjugate roots or zeros α and $\overline{\alpha}$, then the equation or expression can be expressed in the form

$x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha} = 0$ or $a(x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha})$.

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Example 24

a Form a quadratic equation in x that has roots $\alpha = 3 + i\sqrt{5}$ and $\bar{\alpha} = 3 - i\sqrt{5}$, expressing your answer in the form

i $(x - \alpha)(x - \bar{\alpha}) = 0$ ii $x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = 0$.

b Express the quadratic expression $w^2 + 4w + 7$ as a product of its linear factors.

Solution

a i	Write the equation form.	$(x - \alpha)(x - \bar{\alpha}) = 0$
	Substitute roots.	$\left[x-\left(3+i\sqrt{5}\right)\right]\left[x-\left(3-i\sqrt{5}\right)\right]=0$
	Simplify.	$\left(x-3-i\sqrt{5}\right)\left(x-3+i\sqrt{5}\right)=0$
ii	Find the sum of the roots.	$\alpha+\bar{\alpha}=\left(3+i\sqrt{5}\right)+\left(3-i\sqrt{5}\right)$
	Simplify.	$= 6$
	Find the product of the roots.	$\alpha\bar{\alpha}=\left(3+i\sqrt{5}\right)\left(3-i\sqrt{5}\right)$
	Simplify.	$= 14$
	Write the equation form.	$x^2 - (\alpha + \bar{\alpha})x + \alpha\,\bar{\alpha} = 0$
	Substitute sum and product of roots.	$x^2 - 6x + 14 = 0$
b	Write the equation.	$w^2 + 4w + 7 = 0$
	Complete the square.	$(w^2 + 4w + 4) + 3 = 0$
	Write as a difference of squares.	$(w+2)^2-\left(i\sqrt{3}\right)^2=0$
	Factorise as a difference of squares.	$\left(w+2-i\sqrt{3}\right)\left(w+2+i\sqrt{3}\right)=0$

EXERCISE 10.09 Quadratic equations

Concepts and techniques

WS Complex numbers

1 Example 22 Find the solutions to the following real quadratic equations.
a $z^2 + 2z + 4 = 0$ b $z^2 - 2z - 4 = 0$ c $z^2 + 4z + 8 = 0$ d $3z^2 + 2z + 8 = 0$

2 Find the complex solutions to the real quadratic equation $(x + 1)^2 + 9 = 0$.

3 Solve $(x + 1)^2 = 1 - 2(x + 3)^2$ over the complex plane.

4 Example 23 Show by substitution that $\beta = -5 - i$ is a root of the real quadratic equation $z^2 + 10z + 26 = 0$. State the other root.

5 Find the unknown real coefficient if $2 - i$ is a solution to the equation $z^2 + mz + 5 = 0$. State the other solution.

6 Example 24 Form quadratic equations for which the following complex conjugates are roots. Give your answer in the form

i $(x-\alpha)(x-\bar{\alpha})=0$ ii $x^2-(\alpha+\bar{\alpha})x+\alpha\bar{\alpha}=0$

a $1 \pm 2i$ b $\sqrt{3} \pm i$ c $-4 \pm i\sqrt{2}$ d $\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$

e $-5 \pm 6i$ f $\frac{\sqrt{3}}{2} \pm \frac{i}{2}$ g $-\frac{3}{5} \pm i\frac{2}{5}$ h $-\frac{1}{\sqrt{3}} \pm i\frac{\sqrt{2}}{\sqrt{3}}$

7 Find the unknown integer coefficients if the roots below are solutions to the equation $kz^2 + nz + p = 0$.

a $2 \pm i$ b $1 \pm i\sqrt{3}$ c $\frac{-1 \pm i\sqrt{15}}{4}$ d $\frac{5}{13} \pm i\frac{12}{13}$

8 Express each of the quadratic expressions as a product of its linear factors.

a $z^2 - 2z + 2$ b $z^2 - 2z + 6$ c $z^2 + 4z + 5$ d $z^2 + z + 1$.

Reasoning and communication

9 The three roots of the equation $z^3 = 1$ are called the roots of unity. By fully factorising the expression $z^3 - 1$, show that the equation $z^3 = 1$ has one real root and two complex roots which are conjugates.

10 If ω and $\bar{\omega}$ are the complex roots of the equation $z^3 = 1$, show that

a $\bar{\omega} = \omega^2$ b $|\omega| = |\bar{\omega}| = |\omega^2| = 1$

11 Plot the solutions to $z^3 = 1$ as vectors on the complex plane. What do you notice?

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CHAPTER SUMMARY
COMPLEX NUMBERS 10

- The **imaginary number *i*** is defined as $i = \sqrt{-1}$.
- The powers of i make a pattern that repeats each four numbers: $i, -1, -i, 1, i, -1, -i, 1, \ldots$
- A complex number is a number that can be written in the form $a + bi$, where a and b are real numbers.
- A complex number is often denoted by the letter z, so $z = a + bi$.
- The real part of $z = a + bi$ is denoted by $\text{Re}(z)$ and $\text{Re}(z) = a$
- The imaginary part of $z = a + bi$ is denoted by $\text{Im}(z)$ and $\text{Im}(z) = b$.
- If $\text{Re}(z) = 0$, then we say that z is *purely imaginary.*
- If $\text{Im}(z) = 0$, then we say that z is *purely real* or just *real.*
- The set of real numbers is a subset of the set of complex numbers.
- For a complex number z, where $z = a + bi$ (where a and b are real numbers), the complex conjugate of z is denoted by $\overline{z}$ and $\overline{z} = a - bi$
- The product $z\overline{z}$ is real.
- When dividing two complex numbers, we multiply by the complex conjugate of the denominator. This is called *realising the denominator.*
- The complex number $z = x + yi$ (where $x, y \in \boldsymbol{R}$) can be represented on the *Argand diagram* with a horizontal axis denoted by x or $\text{Re}(z)$ and a vertical axis denoted by y or $\text{Im}(z)$.
- The complex number $z = x + yi$ can be represented by the point $P(x, y)$ (Figure 1) or by the vector $\overrightarrow{OP}$ (Figure 2).

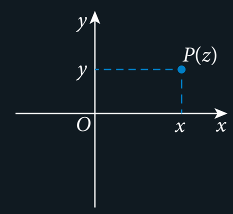


Figure 1

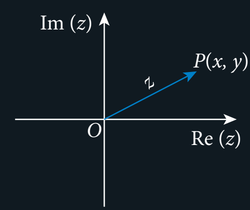
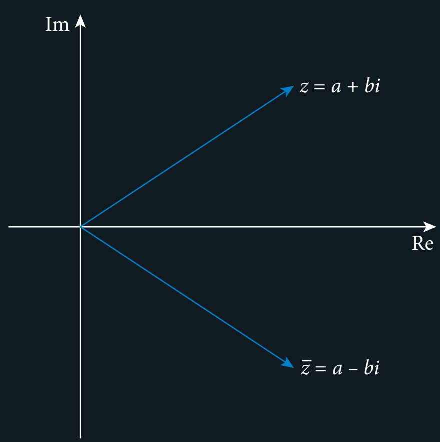
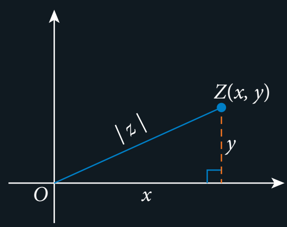


Figure 2

- On the Argand diagram, if $z = a + bi$, then the complex conjugate $\overline{z} = a - bi$ is the reflection of the vector $z = a + bi$ in the x-axis.

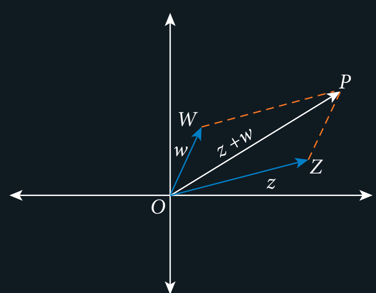


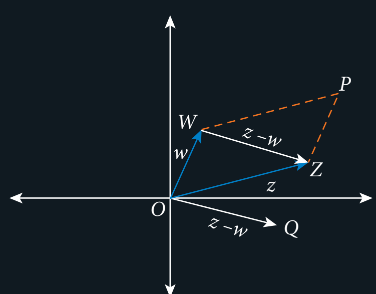
- The length of the vector $z = x + yi$ is called **mod** $\boldsymbol{z}$ or $|z|$.



$|z| = \sqrt{x^2 + y^2}$ or $|z| = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2}$

Given the vectors **z** and **w** with corresponding points Z and W, respectively, the vectors **z** + **w** and **z** − **w** are the diagonal vectors in the parallelogram $OWPZ$.





Point P corresponds with the vector **z** + **w**.

Point Q corresponds with the vector **z** − **w**.

- The complex numbers form a field.
- The following properties hold $\forall z \in \boldsymbol{C}$:
  - a $|z|^2 = z\bar{z}$
  - b $|z|^2 = |z^2|$
  - c $|z_1 z_2| = |z_1||z_2|$
  - d $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - e $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
  - f $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- $z + \bar{z}$ is real $\forall z \in \boldsymbol{C}$.
- $z\bar{z}$ is real $\forall z \in \boldsymbol{C}$.
- If a quadratic equation $ax^2 + bx + c = 0$ has real coefficients a, b and c, then if $\alpha = n + mi$ is a root (n and m are real), then its conjugate $\bar{\alpha} = n - mi$ will also be a root.
- If a quadratic equation $ax^2 + bx + c = 0$ has real coefficients a, b and c, then it is called a *real* quadratic equation. The corresponding expression $ax^2 + bx + c$ is called a *real* quadratic expression.
- If a real quadratic equation has complex conjugate roots α and $\bar{\alpha}$, then the equation can be expressed in the form $(x - \alpha)(x - \bar{\alpha}) = 0$ or $x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = 0$.

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CHAPTER REVIEW COMPLEX NUMBERS 10

Multiple choice

1 Example 6, 10 If $z = 1 - 3i$ and $w = 2 - i$, then $\overline{z} + 2w =$

A $2 + 3i$ B $5 + 5i$ C $5 + i$ D $3 + i$ E $5 - 5i$

2 Example 10 If $z = 1 - 3i$ and $w = 2 - i$, then $zw =$

A $-1 - 7i$ B $-1 - 5i$ C $4 - 7i$ D $4 - 5i$ E 5

3 Example 3 The value of i^{107} is

A i B $-i$ C -1 D 1 E $\sqrt{-107}$

4 Example 16 If $z = \sqrt{2} + i\sqrt{2}$, then $\frac{1}{|z|} =$

A 4 B $\frac{1}{4}$ C 2 D $\frac{1}{2}$ E $\frac{1}{2\sqrt{2}}$

5 Example 21 If z and w are complex numbers, then which of the following is NOT true?

A $\overline{z+w} = \overline{z} + \overline{w}$ B $\overline{z \times w} = \overline{z} \times \overline{w}$ C $|zw| = z\overline{w}$

D $|zw| = |z||w|$ E $z\overline{z} = |z|^2$

Short answer

6 Example 1 Write each number in terms of i.

a $\sqrt{-81}$ b $\sqrt{-18}$ c $\sqrt{-48}$ d $\sqrt{-216}$

7 Example 2 Solve the following equations.

a $z^2 = -36$ b $z^2 + 9 = 0$ c $4z^2 + 1 = 0$

8 Example 3 Evaluate each of the following.

a i^3 b i^{200} c i^{49} d i^{590} e $\frac{1}{i^{17}}$

9 Example 4 Write down Re(z) and Im(z) for each complex number.

a $5 - i\sqrt{5}$ b $\frac{1 + i\sqrt{3}}{2}$

c $x + 3yi - 2y + xi$ (assume that x and y are real)

10 Example 5 Solve the following quadratic equations, giving your answers in the form $a + bi$ (assume that a and b are real).

a $x^2 + 4x + 6 = 0$ b $3z^2 - 10z + 12 = 0$ c $w^2 + w + 2 = 0$

11 Example 6 Write the complex conjugate of:

a $-4 + 7i$ b $2 + 5i$

c $2x + y - yi + 3xi$ (assume that x and y are real)

12 Example 7 By first expressing each as a difference of two squares, factorise each quadratic below.

a $x^2 + 4x + 5$ b $z^2 - 10z + 29$ c $w^2 + w + 1$

13 Example 8 Write a quadratic equation in the form $ax^2 + bx + c = 0$ with the following complex roots.

a $3 \pm i$ b $1 \pm i\sqrt{6}$ c $\dfrac{2 \pm i\sqrt{3}}{2}$

14 Example 9 Find the value of real numbers x and y if $2x + y + i(x - 2y) = 4 + 7i$.

15 Example 10 Simplify each expression below, giving your answers in the form $a + bi$ (assume that a and b are real).

a $2 + 8i - 4 - 6i$ b $(4 - i)(7 + 3i)$ c $(3 - 2i)^2$

16 Example 11 Simplify by realising the denominator, giving your answers in the form $a + bi$ (assume that a and b are real).

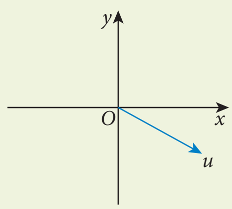
a $\dfrac{1+i}{2-i}$ b $\dfrac{1}{\sqrt{2}+i}$ c $\dfrac{2}{1+i} - \dfrac{2}{1-i}$

17 Example 12 Represent each number as a point on an Argand diagram.

a $z = 2 + i$ b $w = -3 - 2i$ c $v = \dfrac{4}{i}$

18 Example 13 Plot the complex number $z = -1 + 2i$ and its complex conjugate $\bar{z}$ as vectors on an Argand diagram.

19 Example 14 The complex number u is shown as a vector on the Argand diagram to the right. Copy the diagram and plot its conjugate $\bar{u}$.



20 Example 15 For each complex number z find $|z|$.

a $z = -2 + 3i$ b $z = \dfrac{1 + i\sqrt{3}}{4}$ c $z = \dfrac{1}{\sqrt{3} - 2i}$

21 Example 16 Write $|(x - y) - i(2x + y)|$ in simplest form (assume that x and y are real).

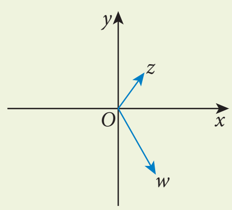
22 Example 17 Plot the complex numbers $z = 3 + i$ and $w = -1 + 2i$ as vectors on an Argand diagram. Hence plot the vector $z + w$.

23 Example 18 Given the complex number $z = 2 - 4i$,

a state the points U and V that represent $\frac{1}{2}z$ and $-z$

b plot U and V on an Argand diagram.

24 Example 19 The complex numbers z and w are shown as vectors on an Argand diagram. Hence plot the point Q that represents the vector $z - w$.



25 Example 20 Find the multiplicative inverse of each complex number, giving your answer in the form $a + bi$ (assume that a and b are real).

a $1 - i\sqrt{2}$ b $2i^3$ c $\dfrac{1}{3 - 4i}$

26 Example 21 Prove that for two complex numbers $z = a + bi$ and $w = c + di$ (assume that a, b, c and d are real):

a $\overline{z + w} = \overline{z} + \overline{w}$ b $\left|\overline{zw}\right| = \left|\overline{z}\right|\left|\overline{w}\right|$

27 Example 22 Solve the following quadratic equations, classifying the roots as real, imaginary or complex.

a $x^2 + 4 = 0$ b $3z^2 - 10z + 3 = 0$ c $w^2 - 2w + 2 = 0$

28 Example 23 Show that $4 - i$ is a root of the equation $z^2 - 8z + 17 = 0$. State the other root.

29 Example 24 Form the quadratic equation that has complex conjugate roots $-3 \pm i\sqrt{2}$, giving your answer in the form $(z - \alpha)(z - \overline{\alpha}) = 0$.

30 Example 10, 11 Evaluate the following.

a $(5 - 2i)(6 + 7i)$ b $|6 - 8i|$ c $\dfrac{\sqrt{2} + i}{\sqrt{2} - i}$ d $\overline{3 - 4i}$

31 Example 4, 9, 15 Consider the complex numbers $u = \sqrt{3} - i$ and $v = -1 - i\sqrt{3}$. Find:

a $u + v$ b $\text{Re}(u + v)$ c $\text{Im}(u + v)$ d $|u + v|$

32 Example 13 If $z = 3 + 2i$, sketch the following as vectors on Argand diagrams.

a z b $\overline{z}$ c $-z$ d $2z$

33 Example 22 Solve each equation in the complex plane.

a $z^2 + 25 = 0$ b $z^2 - 10z + 34 = 0$ c $w^2 + 3w = 0$
d $2u^2 + 8u + 9 = 0$ e $3z^2 + 7z + 5 = 0$ f $w^4 - 5w^2 - 36 = 0$

34 Example 24 If $-2 \pm 2i$ are the zeros of a quadratic $az^2 + bz + c$, find the quadratic.

Application

Practice quiz

35 Find the solutions to $x^2 + 5 = 3x$.

36 By factorising $z^3 - 8$, solve the equation $z^3 = 8$ in the complex plane.

37 The points V, W and U represent the complex numbers $1 + i$, $4 + 2i$ and $-1 + 3i$ respectively on the complex plane. Find the complex number represented by the point X if $VWXU$ is a parallelogram. What is the complex number represented by the midpoint M of VX?